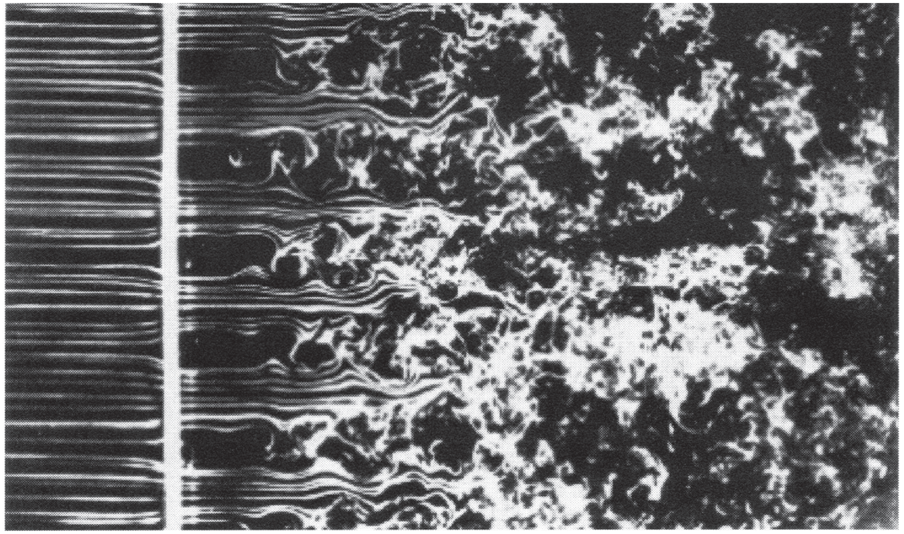
Ambit processes and Turbulence

Jürgen Schmiegel
Ole E. Barndorff-Nielsen
Emil Hedevang

Thiele Centre for Applied Mathematics in Natural Science
University of Aarhus, Denmark

Overview

1. Introduction to turbulence
2. Turbulence statistics
3. Ambit processes and turbulence
4. Model specification
5. Model performance
6. Spatio-temporal models
7. Besides turbulence: tumor growth



Turbulent flows are characterized by low momentum diffusion, high momentum convection, and **rapid variation of pressure and velocity in space and time.**

Flow that is not turbulent is called laminar flow. The non-dimensional Reynolds number R characterizes whether flow conditions lead to laminar or turbulent flow. Increasing the Reynolds number increases the turbulent character and the limit of infinite Reynolds number is called the fully developed turbulent state.

Observable: velocity $\vec{v}(\vec{r},t)$

quantities of interest: velocity increments

$$\vec{u}(\vec{r}_0,\vec{r},t) = \vec{v}(\vec{r}_0+\vec{r},t) - \vec{v}(\vec{r}_0,t)$$

energy dissipation

$$\varepsilon(\vec{r},t) \propto \sum_{i,j=1}^{3} \left(\partial_i v_j + \partial_j v_i\right)^2$$

Navier Stokes Equation: incompressible fluid

$$\partial_t \vec{v} + (\vec{v} \cdot \nabla)\vec{v} = -\nabla p + \nu \Delta \vec{v} + \vec{f}$$

$$\nabla \cdot \vec{v} = 0$$

ν : viscosity

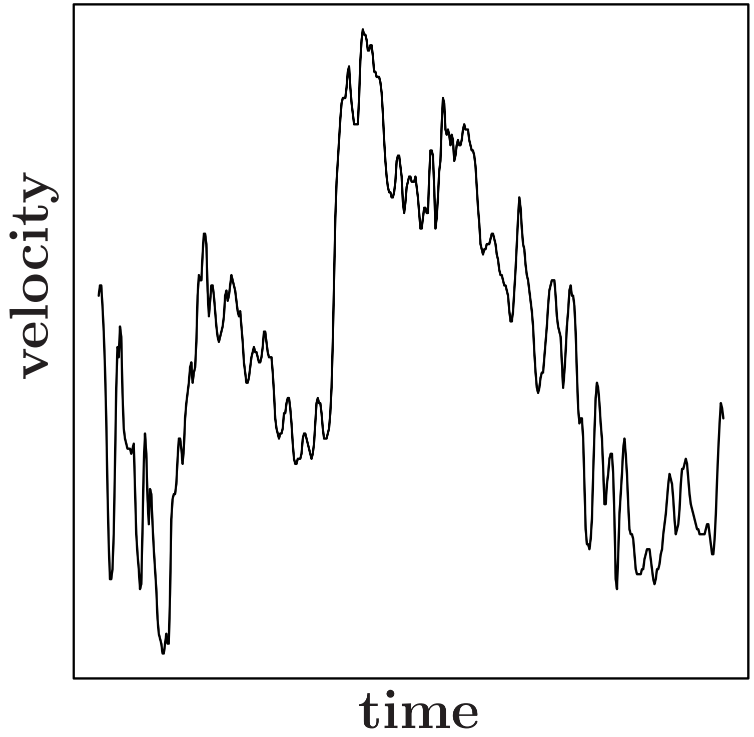
p : pressure

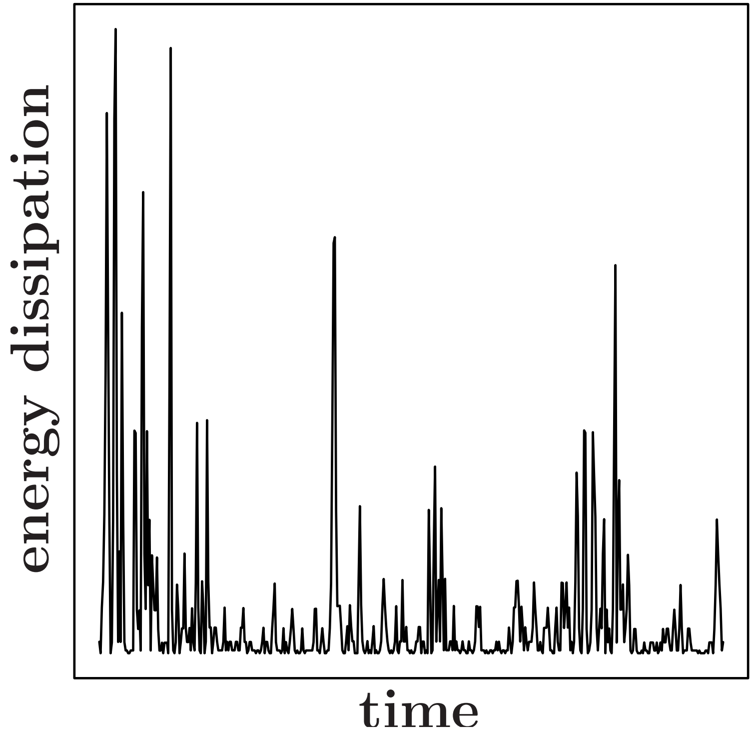
$\vec{f}$: (stochastic) force

Good turbulent data sets:	stationary, isotropic and homogeneous
measured time series:	$v(t) = v_{main}(\vec{r}_0, t)$
velocity increments:	$u(s) = v(s) - v(0)$
(surrogate) energy dissipation:	$\varepsilon(t) \propto \left(\frac{v(t+\Delta t) - v(t)}{\Delta t} \right)^2$

THIELE CENTRE
FOR APPLIED MATHEMATICS IN NATURAL SCIENCE

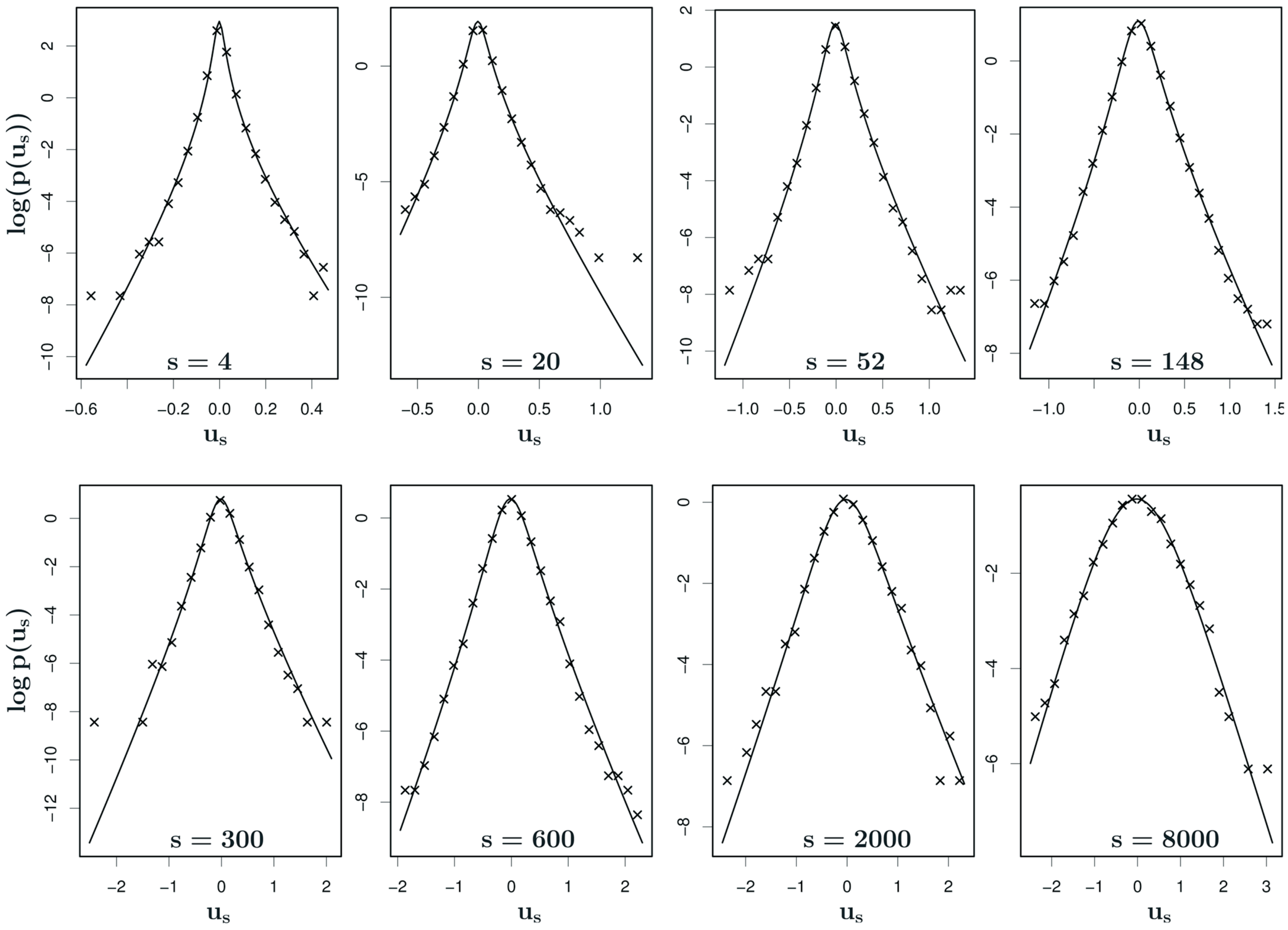
Turbulent time series





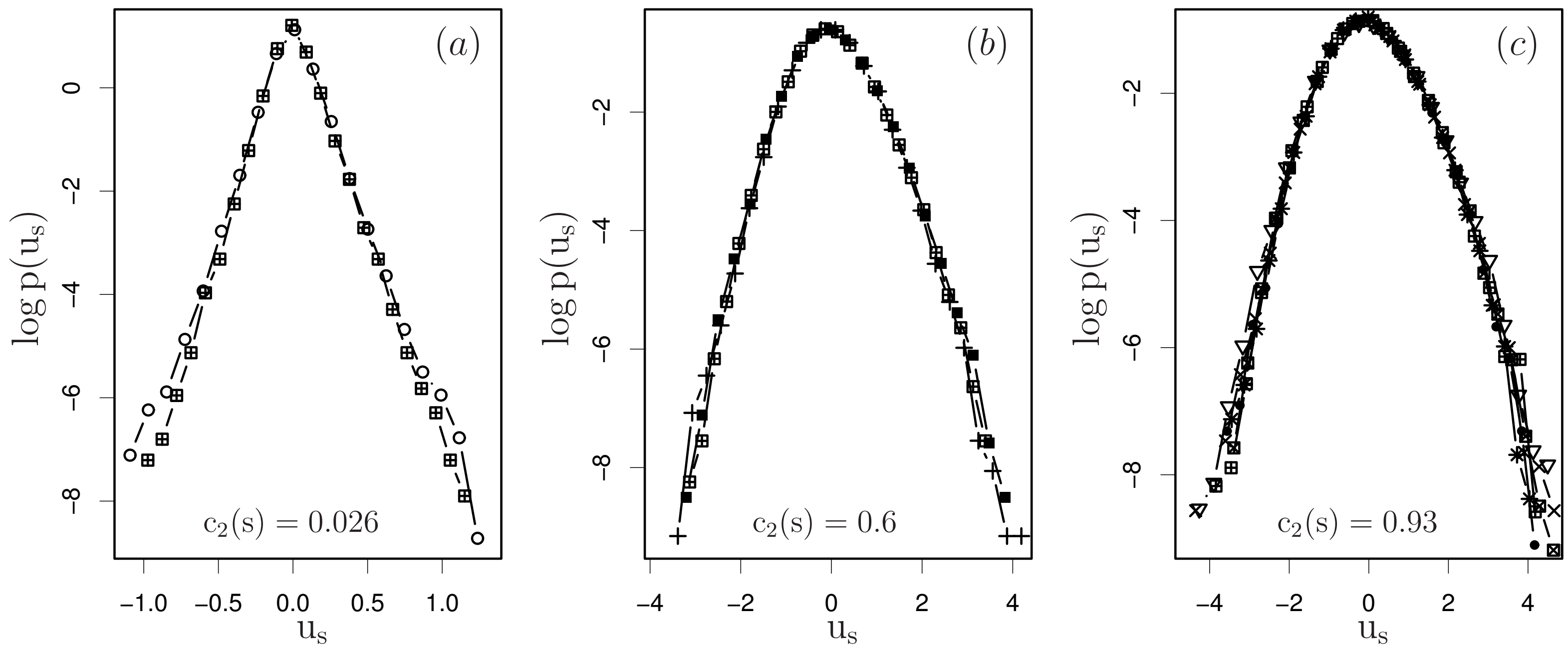
THIELE CENTRE
FOR APPLIED MATHEMATICS IN NATURAL SCIENCE

Aggregational Gaussianity: atmospheric boundary layer



Universality: turbulent data sets (1) and (2)

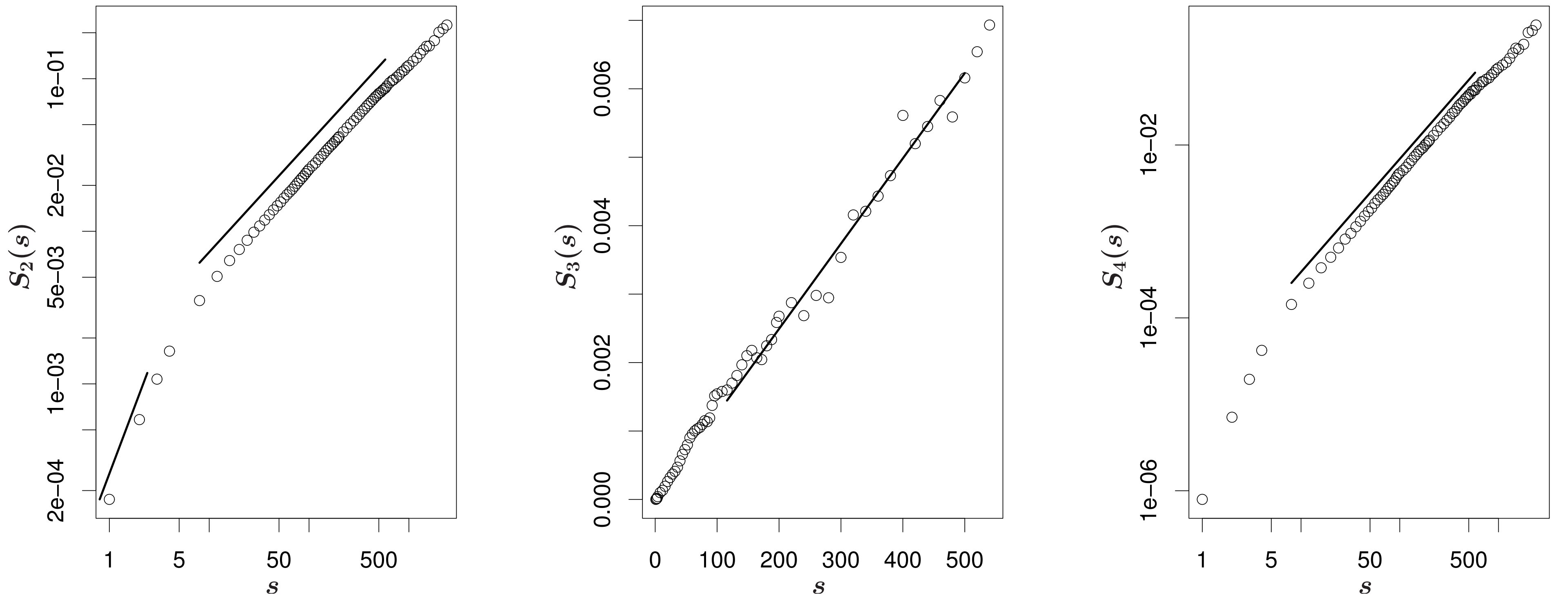
$$u^{(1)}(s_1) \cong u^{(2)}(s_2) \Leftrightarrow \mathrm{Var}\left(u^{(1)}(s_1)\right) = \mathrm{Var}\left(u^{(2)}(s_2)\right)$$



Structure functions: $S_n(s) = E\{u_s^n\} \propto s^{-\tau(n)}$

for s within the inertial range and large Reynolds numbers

atmospheric boundary layer

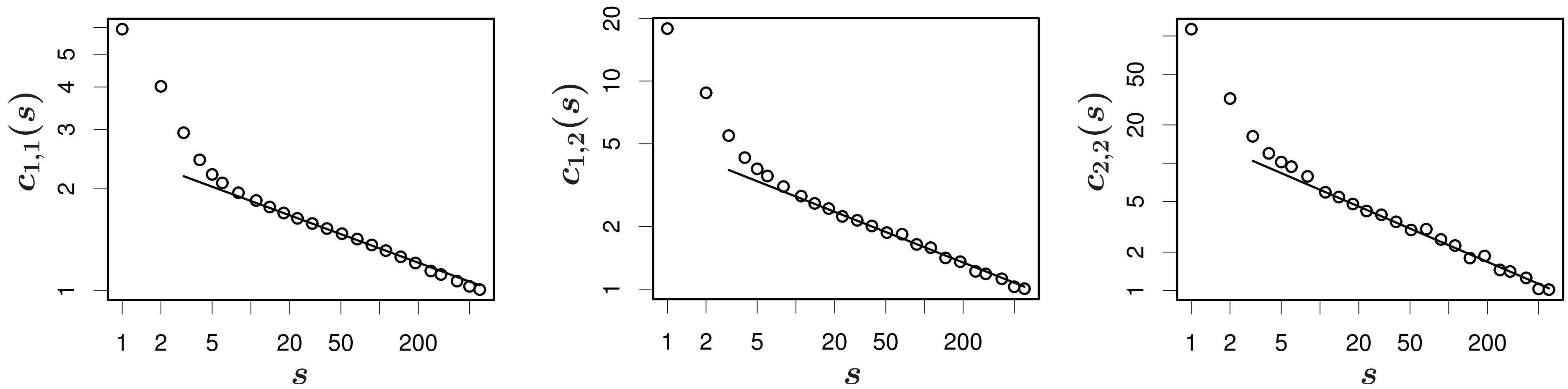


energy dissipation correlators:

$$c_{n_1 n_2}(s) = \frac{E\left\{\varepsilon(0)^{n_1}\,\varepsilon(s)^{n_2}\right\}}{E\left\{\varepsilon(0)^{n_1}\right\}E\left\{\varepsilon(s)^{n_2}\right\}} \propto s^{-\xi(n_1,n_2)}$$

for s within the inertial range and large Reynolds numbers

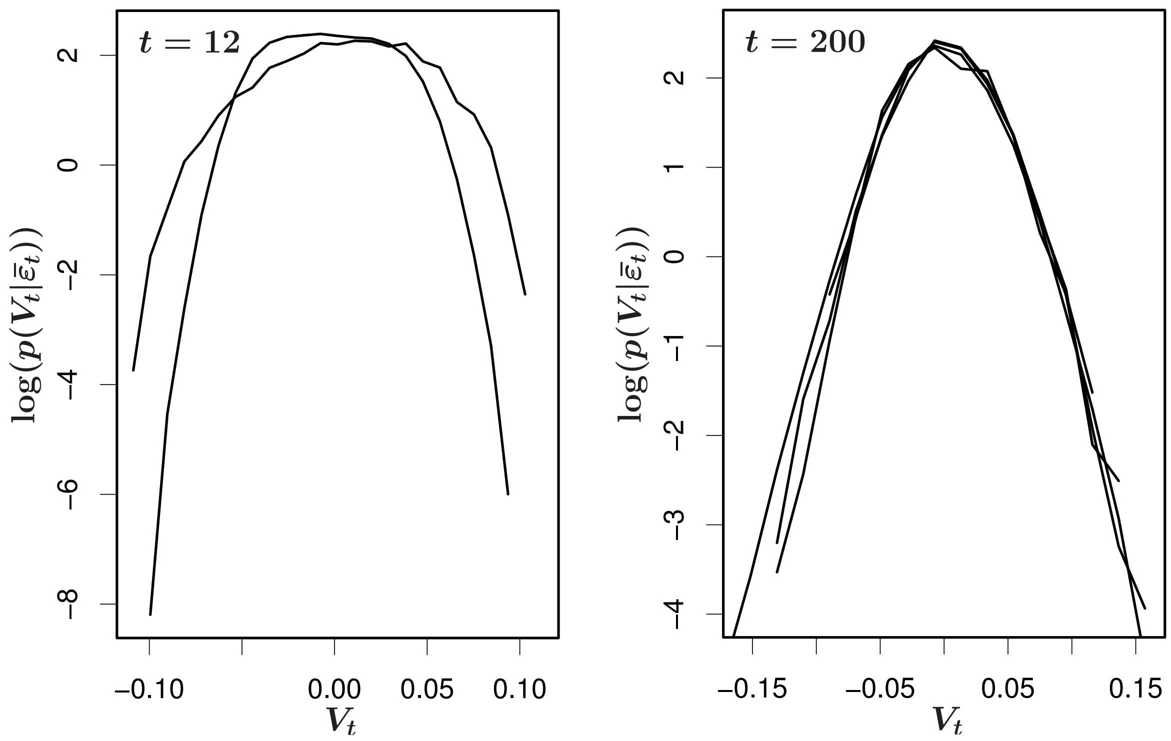
helium jet experiment



THIELE CENTRE FOR APPLIED MATHEMATICS IN NATURAL SCIENCE

Kolmogorov variable: $V_t = \frac{u(t)}{\bar{\varepsilon}_t^{1/3}}$ where $\bar{\varepsilon}_t = \int_0^t \varepsilon_s ds$

atmospheric boundary layer



main hypothesis:

$p(V_t|\bar{\varepsilon}_t)$

does not depend on $\bar{\varepsilon}_t$ for t within the inertial range and large local Reynolds number

$R_t = \frac{t\bar{\varepsilon}_t^{1/3}}{\nu}$

realized quadratic variation:

$$[u_\delta]_t = \sum_{j=1}^{\lfloor t/\delta \rfloor} \left(v(j\delta) - v((j-1)\delta)\right)^2$$

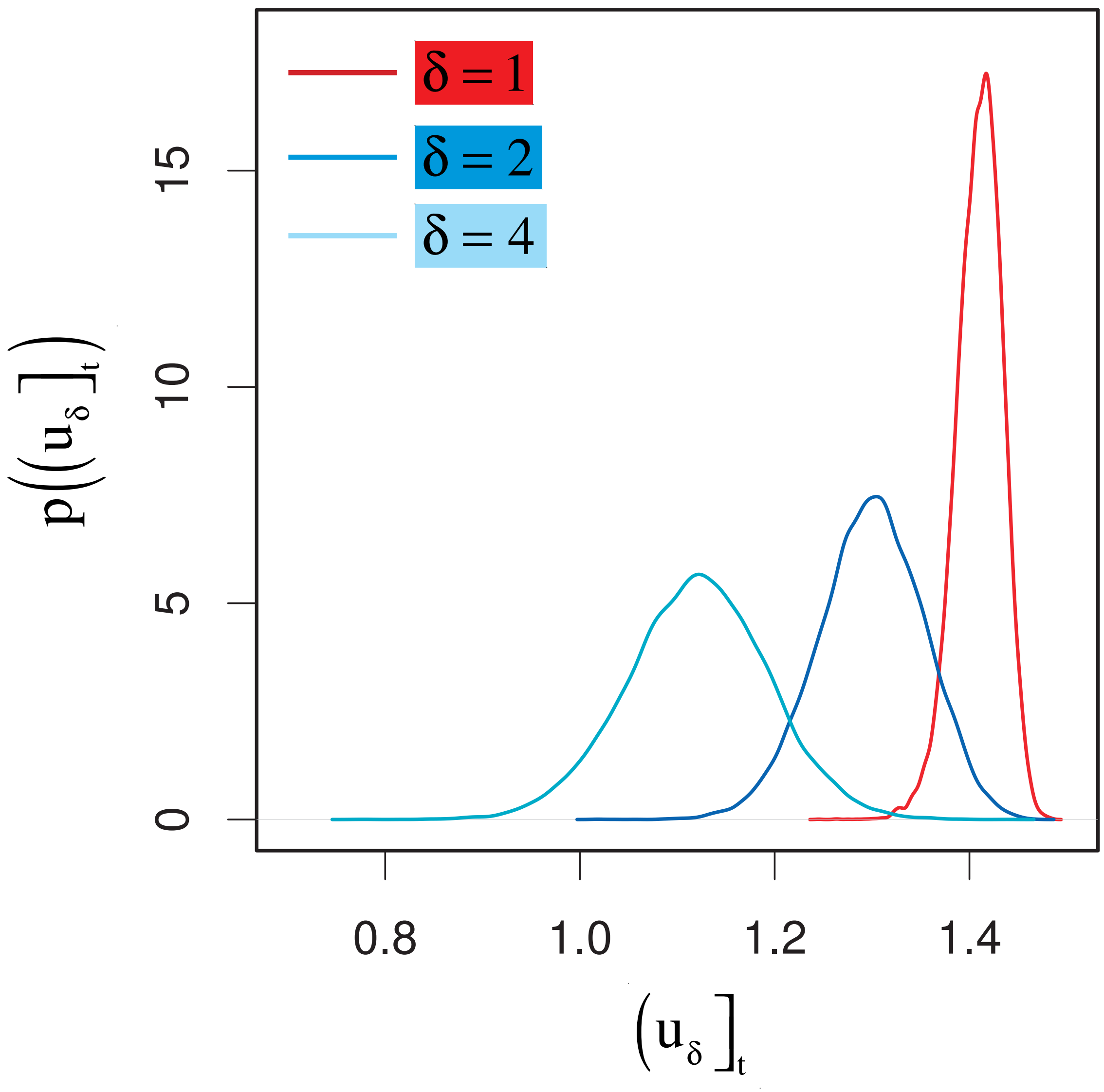
realized bipower variation:

$$[u_\delta]_t^{[1,1]} = \sum_{j=2}^{\lfloor t/\delta \rfloor} \left|v((j-1)\delta) - v((j-2)\delta)\right| \left|v(j\delta) - v((j-1)\delta)\right|$$

realized variation ratio:

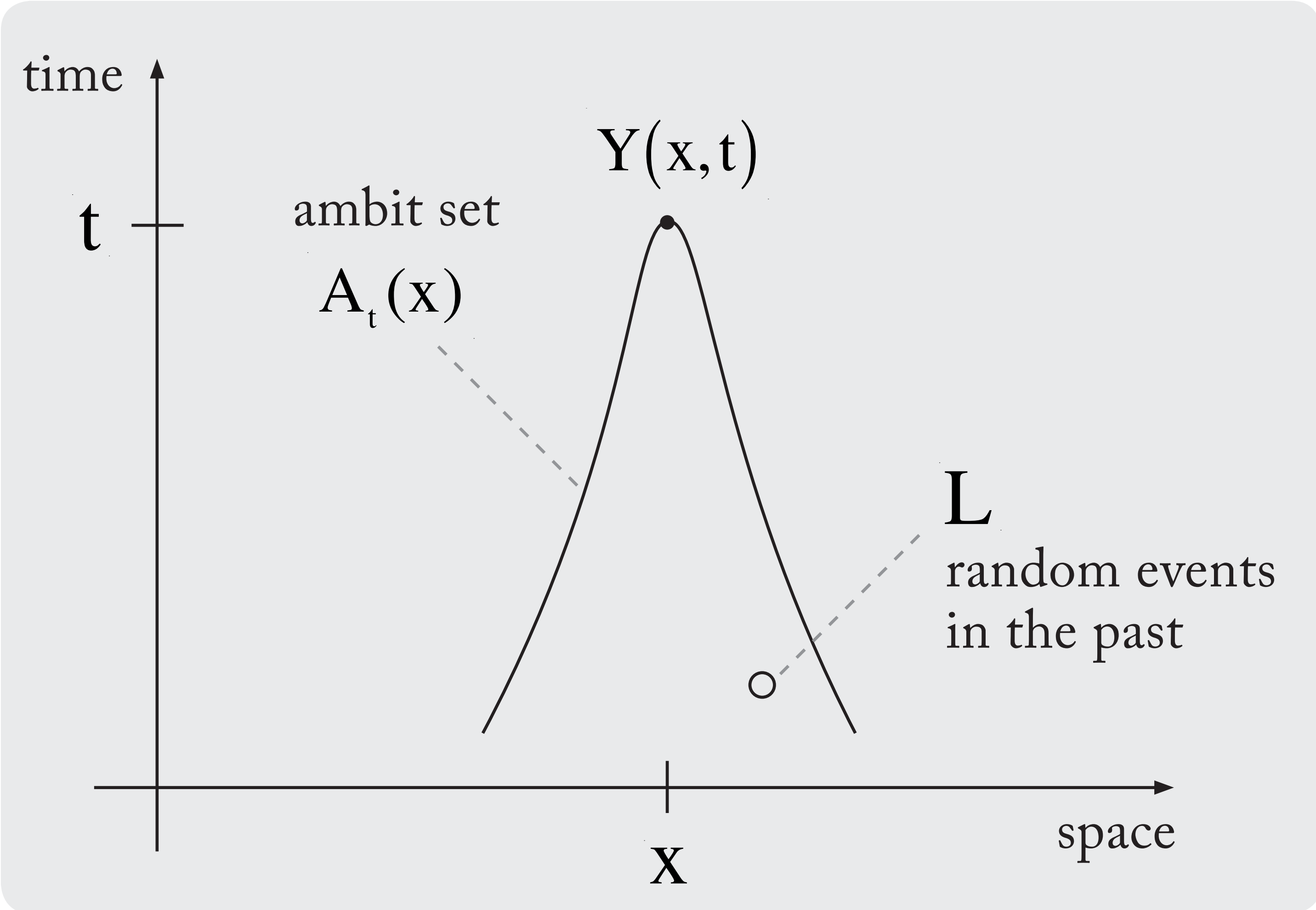
$$(u_\delta]_t \propto \frac{[u_\delta]_t^{[1,1]}}{[u_\delta]_t}$$

helium jet experiment:



THIELE CENTRE FOR APPLIED MATHEMATICS IN NATURAL SCIENCE

intuitive approach: causality cone



ambit process: $$Y(x,t) = \int_{A_t(x)} \cdots dL$$

stochastic intermittency field: $$Y(x,t) = \exp\left\{ \int_{A_t(x)} \cdots dL \right\}$$

more specific: **turbulent velocity field**

$$v(x,t) = \int_{A_t(x)} g\left(t-s;|\rho - x|\right)\sigma_s(\rho)L(ds d\rho) + \int_{B_t(x)} f\left(t-s;|\rho - x|\right)\sigma_s^2 ds d\rho$$

g, f: deterministic functions

$A_t(x), B_t(x) \subset \mathbb{R}^4$: ambit sets

σ^2: intermittency

L: Lévy basis

timewise modelling : **Brownian semistationary processes**

$$v(t) = \int_{-\infty}^{t} g(t-s)\sigma_s dB_s + \beta \int_{-\infty}^{t} g(t-s)\sigma_s^2 ds$$

β : constant

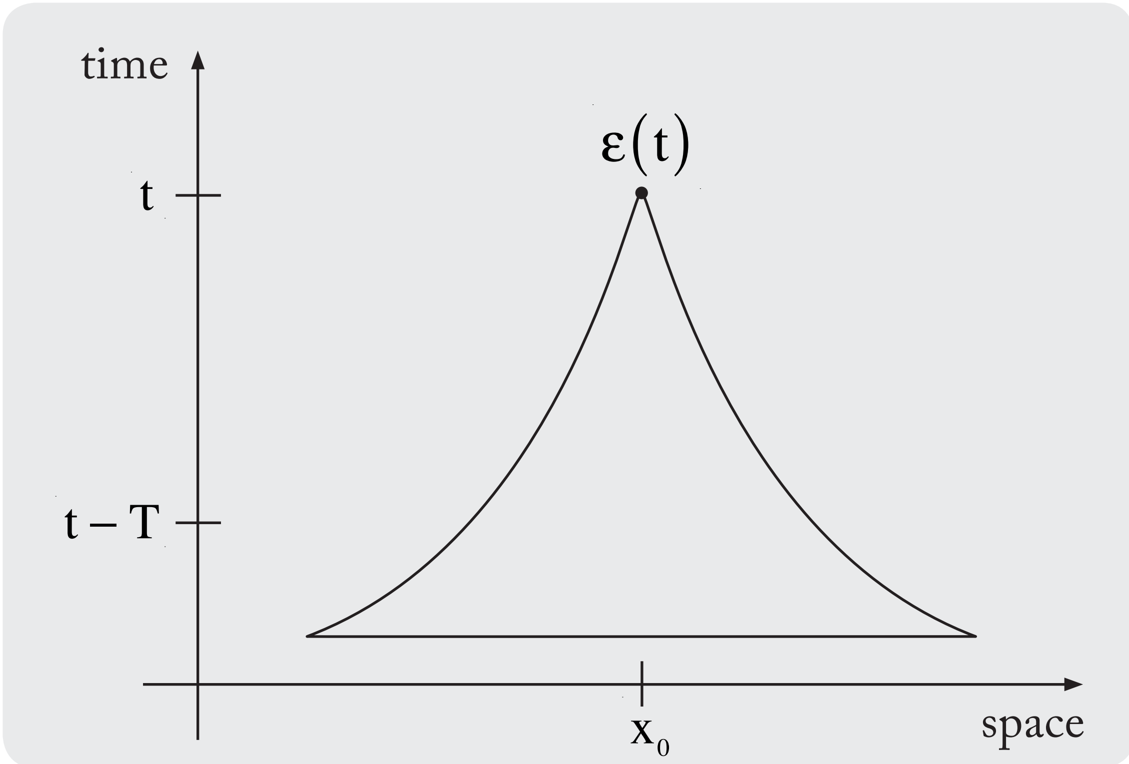
B : Brownian motion

σ^2: plays the role of the energy dissipation

continuous cascade process: stochastic intermittency field

$$\varepsilon(t) = \sigma^2(t) = \exp\left\{ \int_{t-T}^{t} \int_{x_0 - r(t-T+s)}^{x_0 + r(t-T+s)} L(ds\,d\rho) \right\}$$

$$r(t) = \frac{a}{t + t_0}$$



THIELE CENTRE FOR APPLIED MATHEMATICS IN NATURAL SCIENCE

energy dissipation correlators: approximate scaling

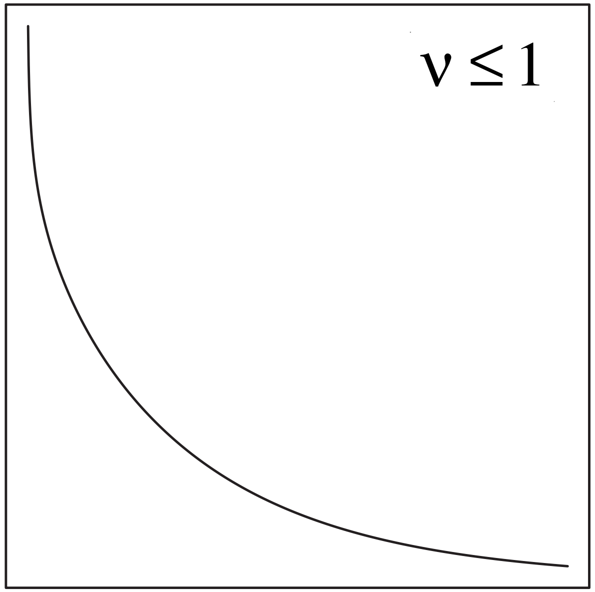
$$c_{n_1 n_2}(s) = \frac{E\left\{\varepsilon(0)^{n_1}\varepsilon(s)^{n_2}\right\}}{E\left\{\varepsilon(0)^{n_1}\right\}E\left\{\varepsilon(s)^{n_2}\right\}} = \exp\left\{\bar{K}[n_1, n_2]\int_s^T 2r(t)dt\right\}$$

$$\propto \left(s + t_0\right)^{-2a\bar{K}[n_1, n_2]}$$

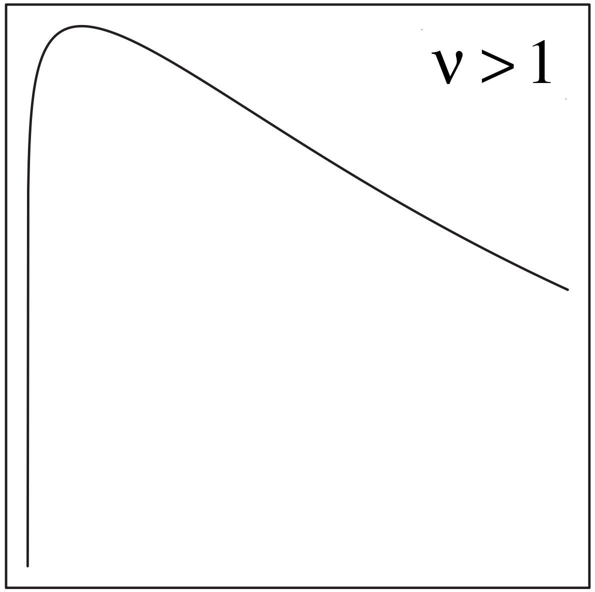
$$v(t) = \int_{-\infty}^{t} g(t-s)\sigma_s \, dB_s + \beta \int_{-\infty}^{t} g(t-s)\sigma_s^2 \, ds$$

$$g(t) = t^{\nu-1} e^{-\lambda t}$$

Model (a)



Model (b)

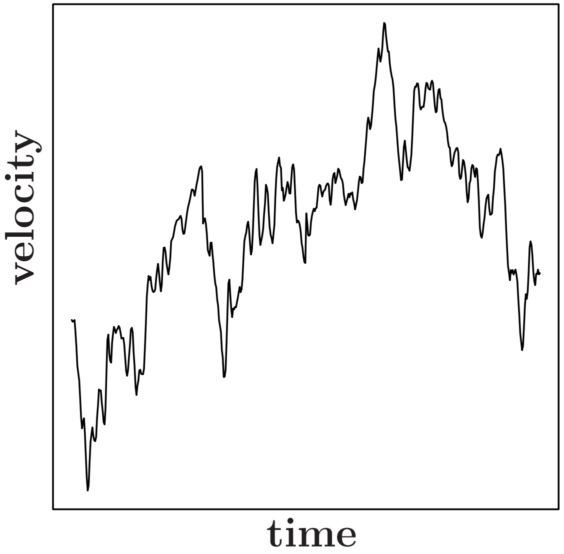


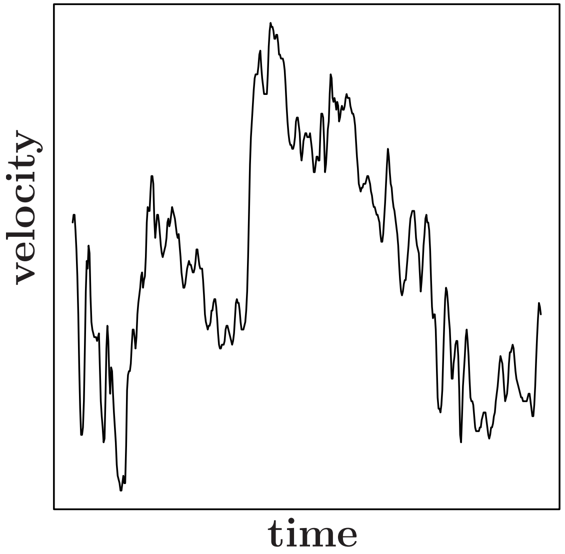
Model performance

both models reproduce:

- scaling of structure functions
- scaling of energy dissipation correlators
- aggregational Gaussianity
- distributions of the velocity increments that are well fitted by NIG-distributions
- conditional independence of the Kolmogorov variable

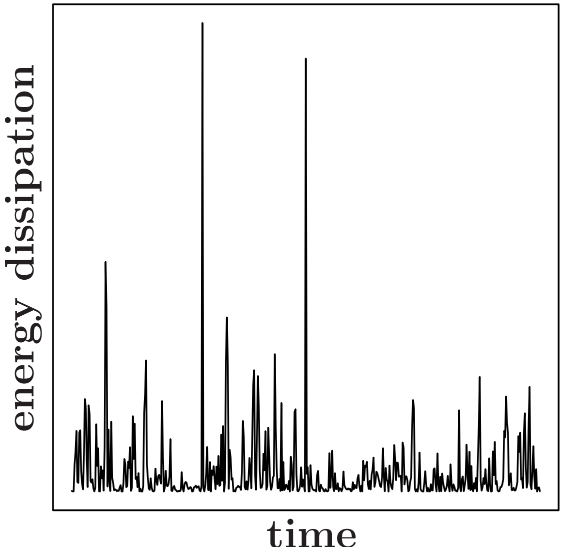
Model performance

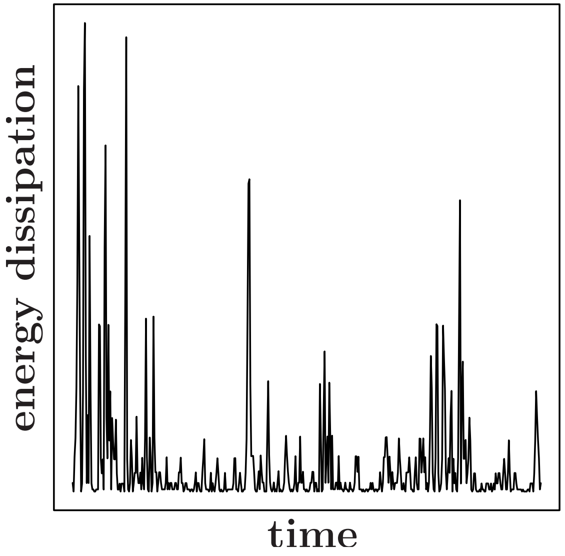




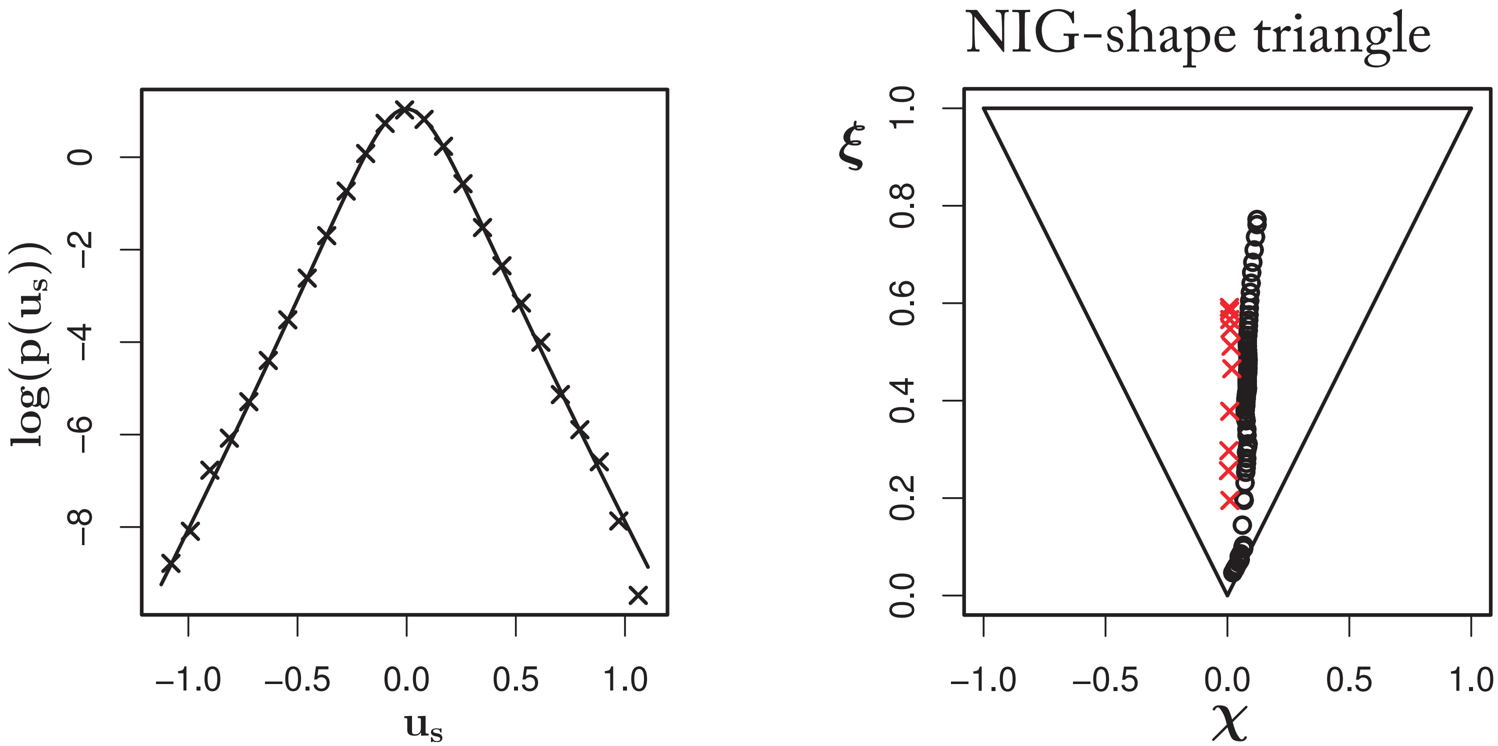
simulation

experiment



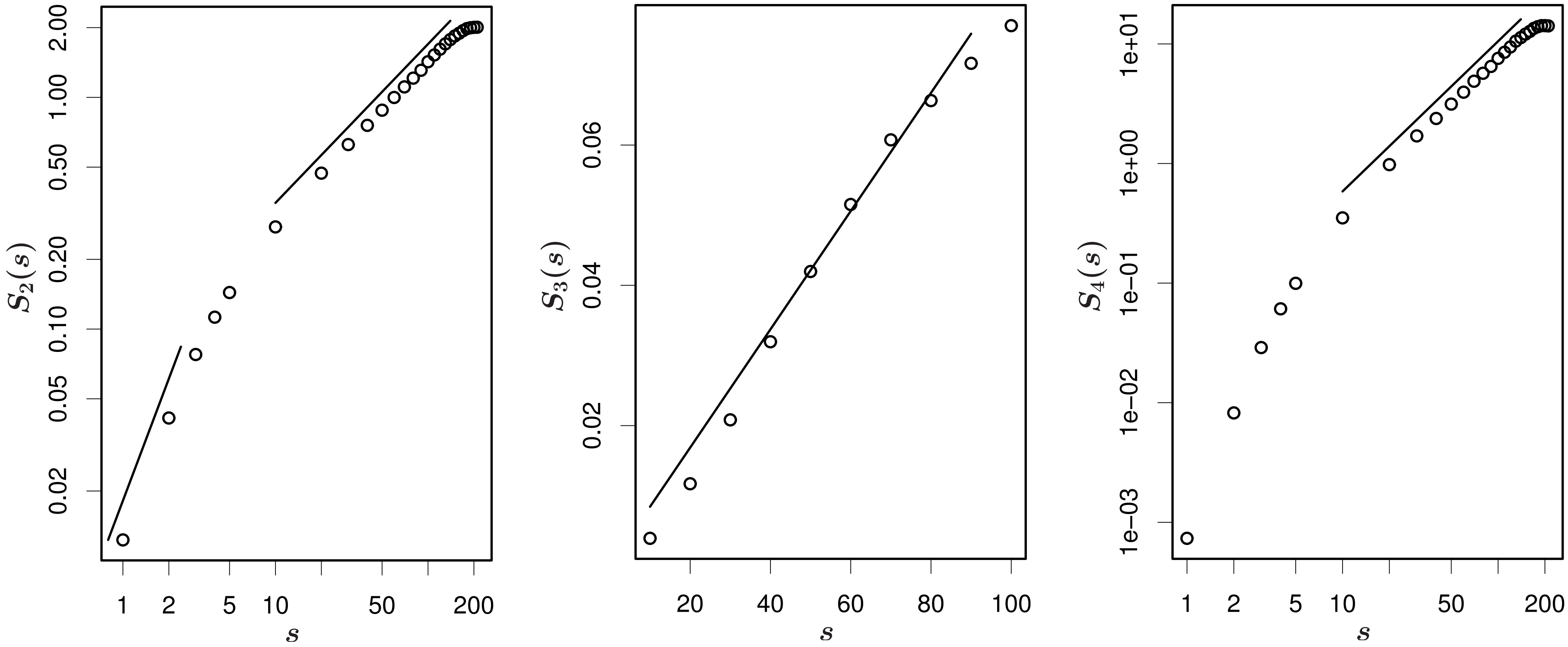


densities of velocity increments:



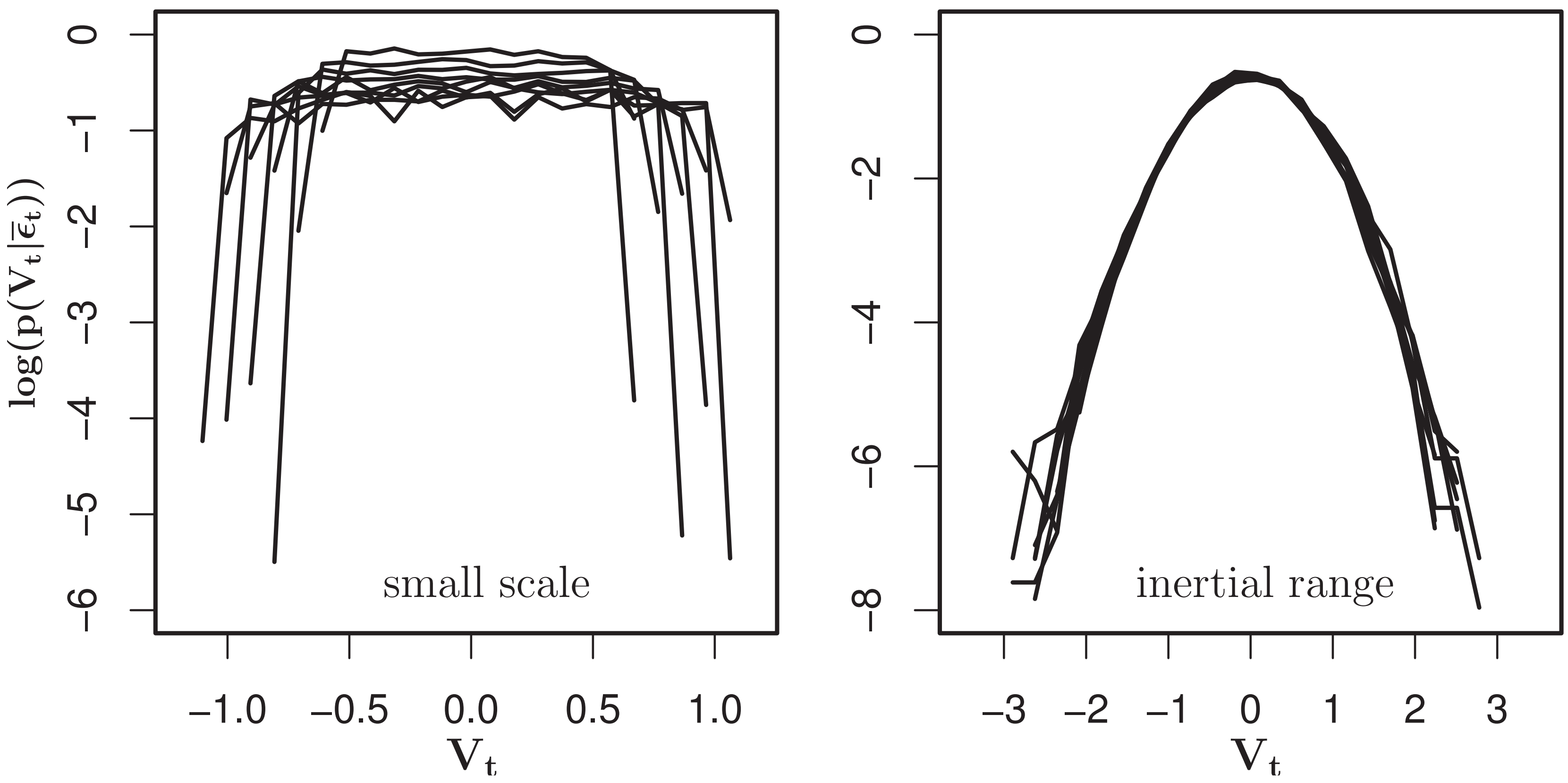
Model performance

scaling of structure functions:

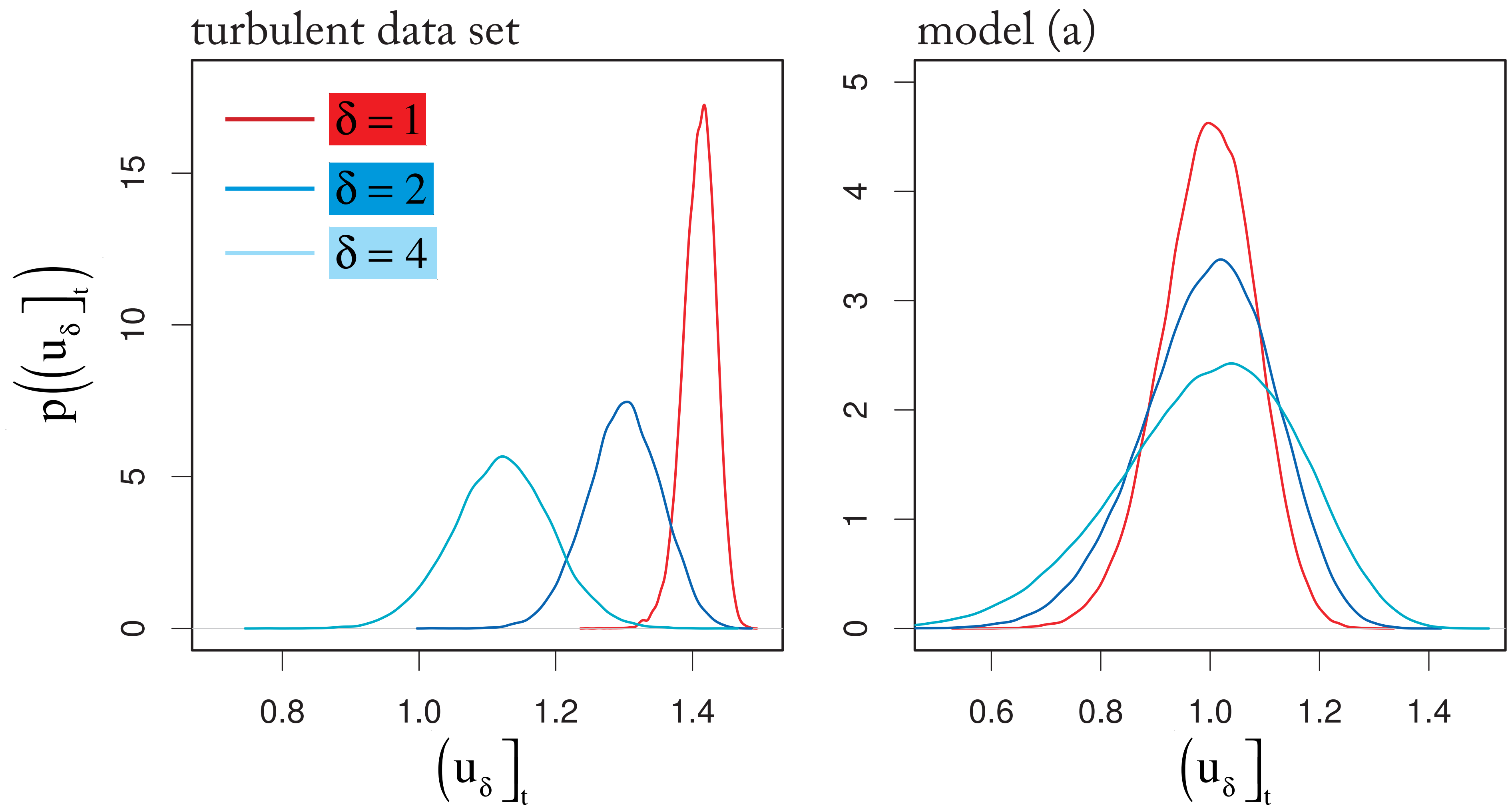


THIELE CENTRE
FOR APPLIED MATHEMATICS IN NATURAL SCIENCE

conditional independence of the Kolmogorov variable:

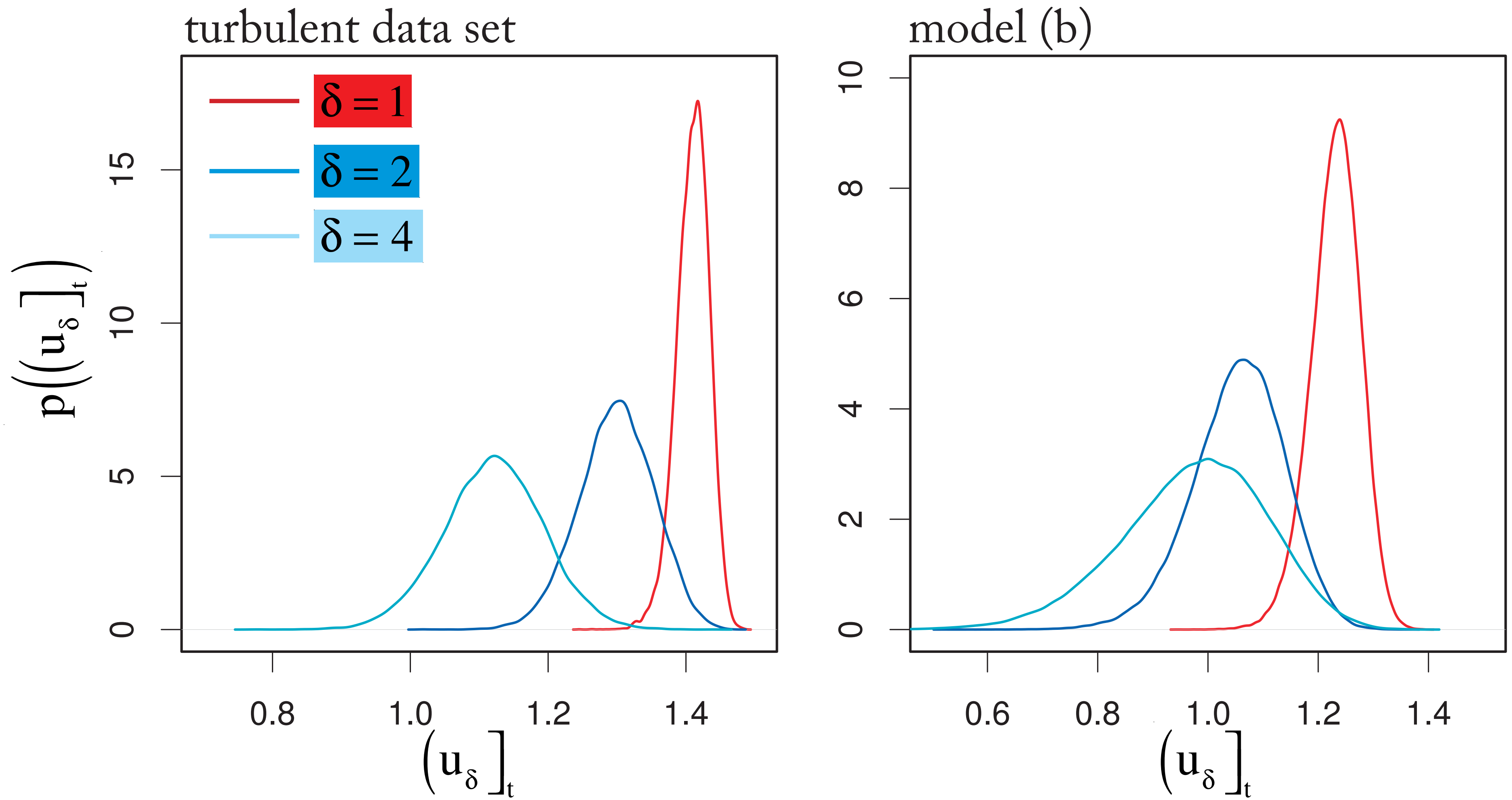


realized variation ratio:



THIELE CENTRE FOR APPLIED MATHEMATICS IN NATURAL SCIENCE

realized variation ratio:

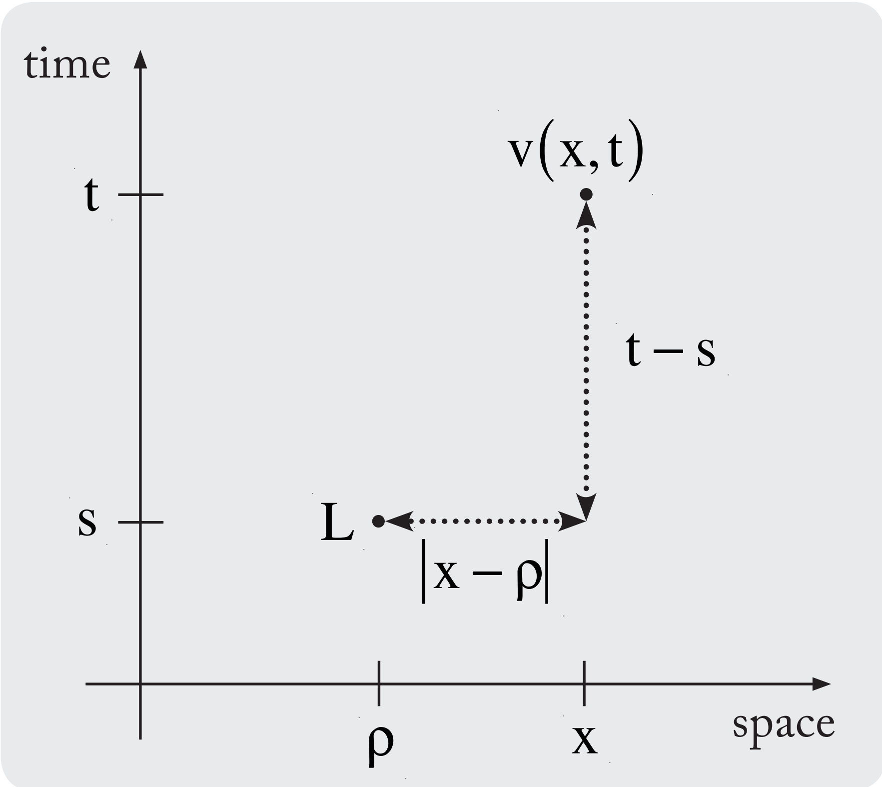


energy dissipation: spatio-temporal continuous cascade process

$$\varepsilon(x,t) = \exp\left\{ \int_{t-T}^{t} \int_{A_s(x)} L(ds\,d\rho) \right\}$$

scaling of energy dissipation correlators in space and time

velocity field: ambit-set in space-time



causality:

random event L at (ρ, s) can influence the velocity v at (x,t) only if $s \leq t$ and if it can reach the point x within the time $t-s$

advection: advection velocity v_a

densitiy fluctuations: speed of sound c_0

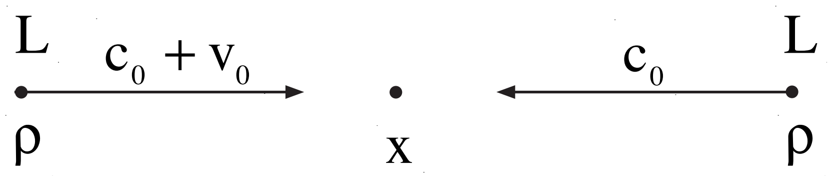
Spatio-temporal models

Consider one spatial dimension x and let $v_0 = E\{v\} > 0$ be the mean velocity.

assumption: $0 \le v_a \le v_0$

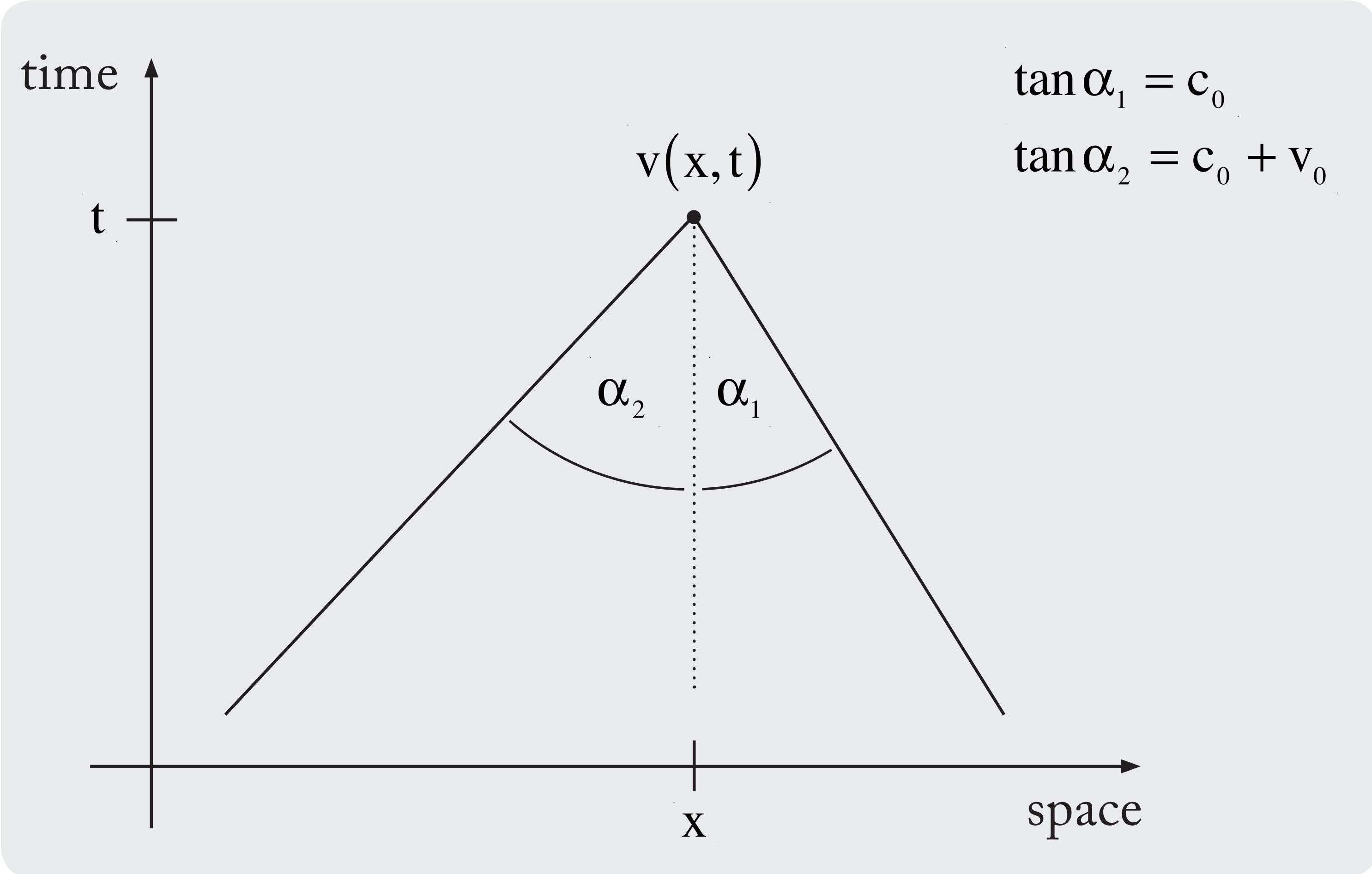
The random event L at (ρ, s) can influence the velocity v at (x, t) only if $s \le t$ and

$$|\rho - x| \le \begin{cases} (c_0 + v_0)(t - s) & \text{for} \quad \rho \le x \\ c_0 (t - s) & \text{for} \quad \rho > x \end{cases}$$



THIELE CENTRE
FOR APPLIED MATHEMATICS IN NATURAL SCIENCE

one spatial dimension: (asymmetric) triangular ambit set



one spatial dimension:

$$v(x,t) = \int_{-\infty}^{t} \int_{x-(c_0+v_0)(t-s)}^{x+c_0(t-s)} g(t-s;|x-\rho|)\sigma(\rho,s)L(ds\,d\rho)$$

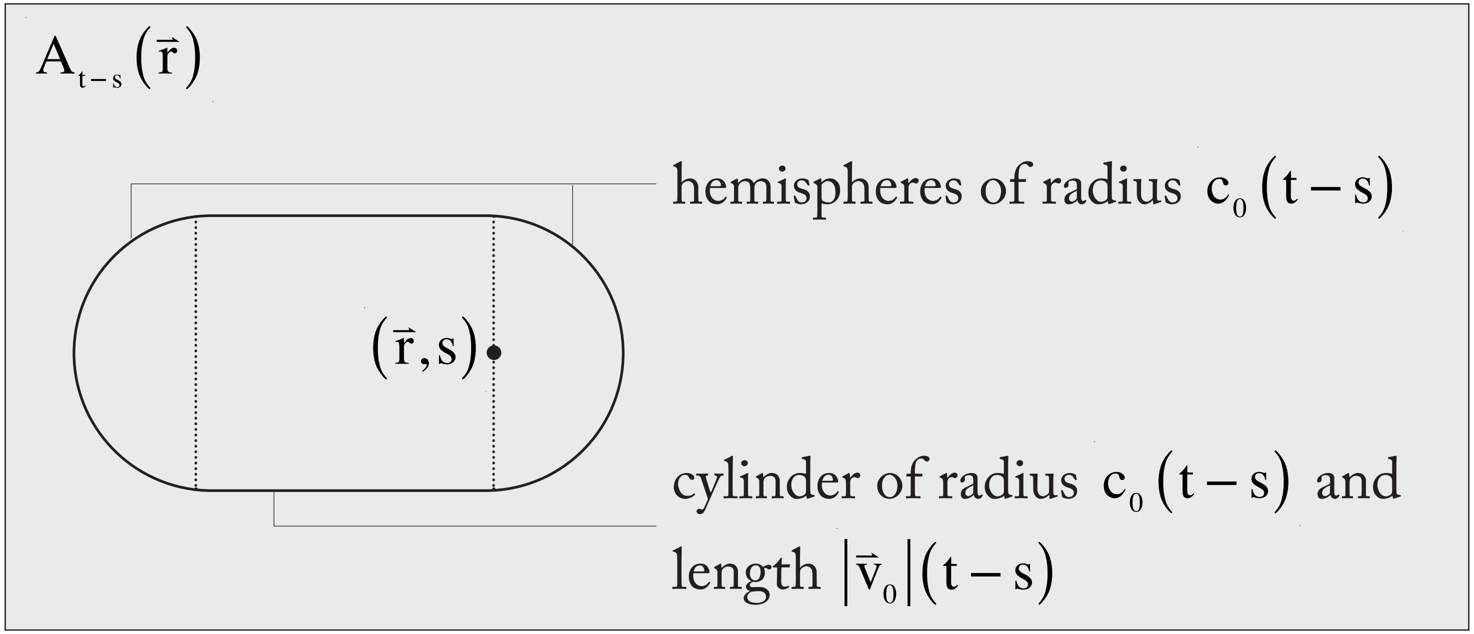
$$+\beta \int_{-\infty}^{t} \int_{x-(c_0+v_0)(t-s)}^{x+c_0(t-s)} g(t-s,|x-\rho|)\sigma^2(\rho,s)\,ds\,d\rho$$

THIELE CENTRE
FOR APPLIED MATHEMATICS IN NATURAL SCIENCE

three spatial dimensions: consider one component v of the velocity vector $\vec{v}$ and let $\vec{v}_0 = E\{\vec{v}\}$.

assumption: advection in direction $\vec{v}_0$ and $0 \le v_a \le |\vec{v}_0|$

conditions for a random event L at $(\vec{\rho}, s)$ to influence $v(\vec{r}, t)$: $(\vec{\rho}, s) \in A_{t-s}(\vec{r})$



three spatial dimensions: one component of the velocity vector

$$v(\vec{r},t)=\int_{-\infty}^{t}\int_{A_{t-s}(\vec{r})} g\left(t-s;|\vec{r}-\vec{\rho}|\right)\sigma\left(\vec{\rho},s\right)L\left(ds\,d\rho_1\,d\rho_2\,d\rho_3\right)$$

$$+\beta\int_{-\infty}^{t}\int_{A_{t-s}(\vec{r})} g\left(t-s;|\vec{r}-\vec{\rho}|\right)\sigma^2\left(\vec{\rho},s\right)ds\,d\rho_1\,d\rho_2\,d\rho_3$$

three spatial dimensions: **full velocity vector** $\vec{v}$

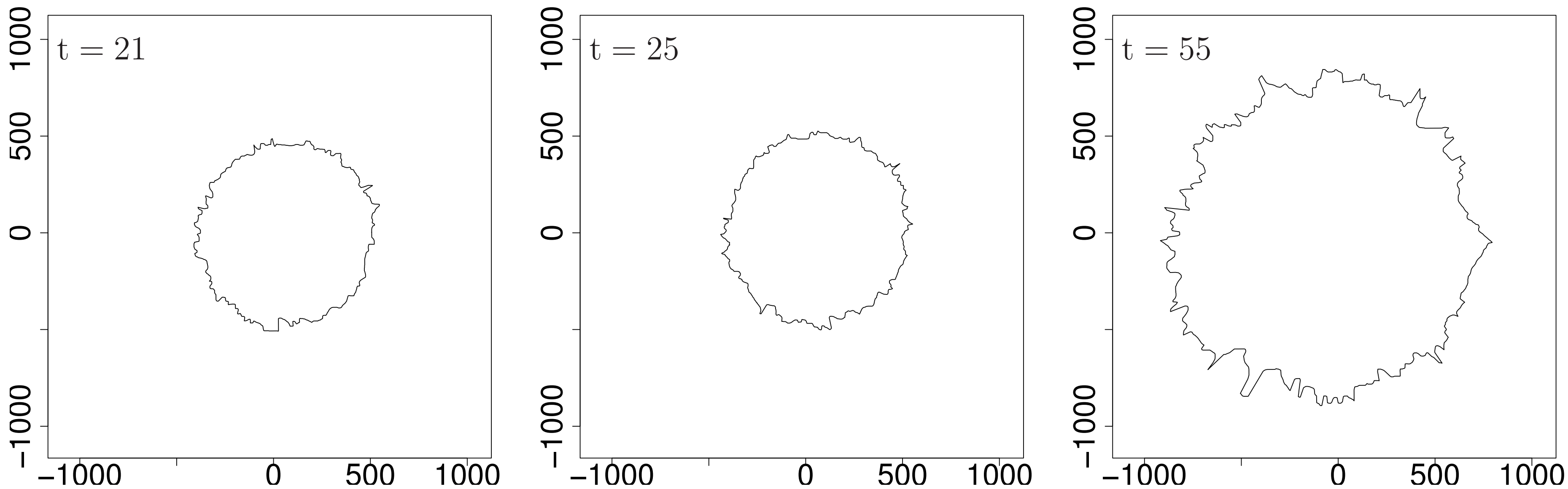
$$\vec{v}(\vec{r},t) = \int_{-\infty}^{t} \int_{A_{t-s}(\vec{r})} g\left(t-s;\left|\vec{r}-\vec{\rho}\right|\right)\sigma(\vec{\rho},s)L\left(ds\,d\rho_1 d\rho_2 d\rho_3\right)$$

$$+\beta \int_{-\infty}^{t} \int_{A_{t-s}(\vec{r})} g\left(t-s;\left|\vec{r}-\vec{\rho}\right|\right)\sigma^2(\vec{\rho},s)\,ds\,d\rho_1 d\rho_2 d\rho_3$$

where $g \cdot \sigma \cdot L$ and $\beta \cdot g \cdot \sigma^2$ are vectors

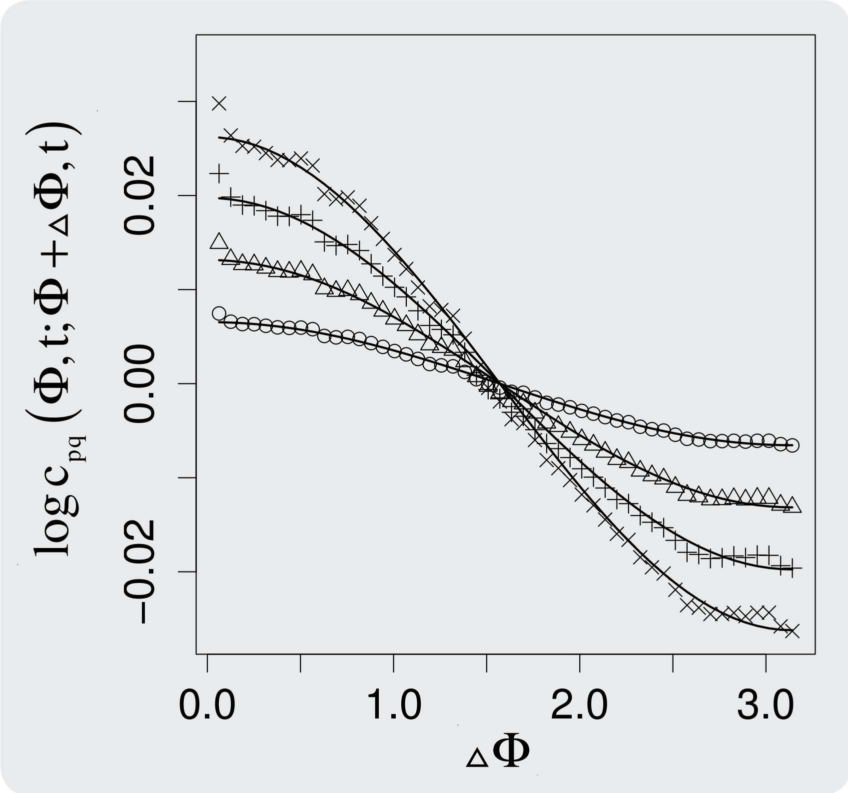
Besides turbulence: tumor growth

The star shaped approximation of a growing brain tumor in vitro at various times t is described by a unique radius function $R_t(\Phi)$.



empirically observed equal time correlators of order (p, q)

$$c_{pq}(\Phi, t; \Phi + \Delta\Phi, t) = \frac{E\{R_t(\Phi)^p R_t(\Phi + \Delta\Phi)^q\}}{E\{R_t(\Phi)^p\} E\{R_t(\Phi + \Delta\Phi)^q\}}$$



Besides turbulence: tumor growth

model for the normalized radius function

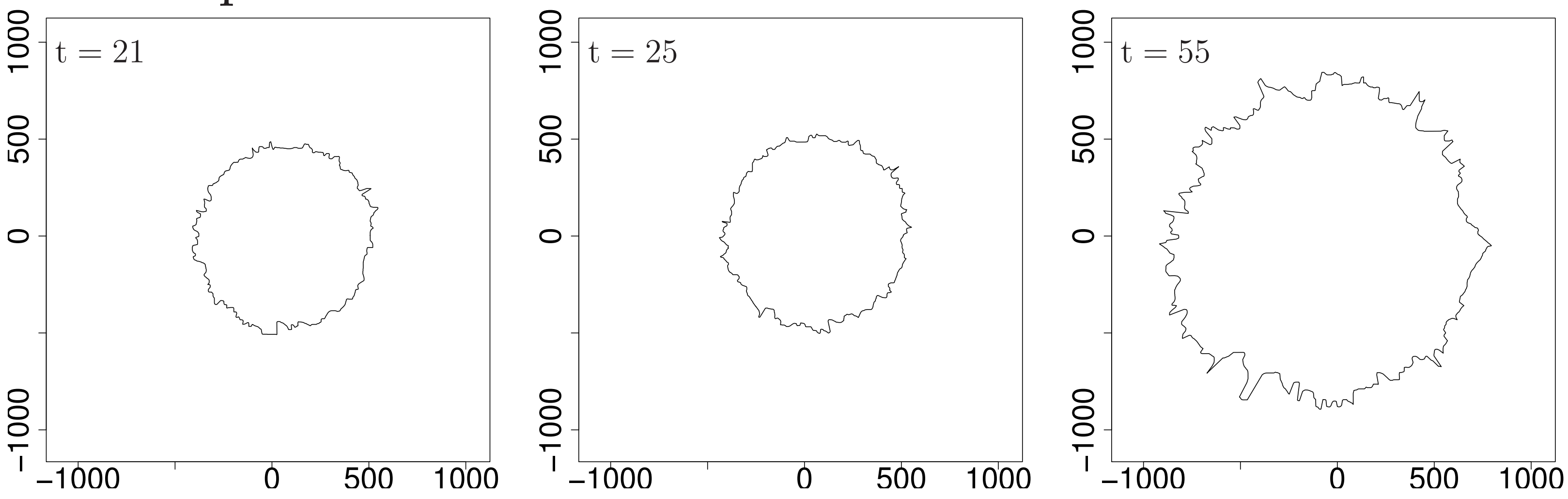
$$r_t(\Phi) = \frac{R_t(\Phi)}{E\{R_t(\Phi)\}}$$

stochastic intermittency field

$$r_t(\Phi) = \exp\left\{ a(t) \int_{A_t^{(1)}(\Phi)} \cos(\Phi - \Phi') L(dt' d\Phi') \right\} + h(t) \int_{A_t^{(2)}(\Phi)} L(dt' d\Phi')$$

THIELE CENTRE
FOR APPLIED MATHEMATICS IN NATURAL SCIENCE

star shaped tumor



simulation