A Milstein-type scheme without Lévy area terms for SDEs driven by fractional Brownian motion

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Ambit Processes Sandbjerg

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SDEs driven by FBM

$$(SDE) \quad dX_t = \sum_{j=0}^{m} \sigma^{(j)}(X_t) dB_t^{(j)}, \quad t \in [0,1], \qquad X_0 = x_0 \in \mathbb{R}^d$$

where

- $\sigma^{(0)}, \sigma^{(1)}, \ldots, \sigma^{(m)} : \mathbb{R}^d \to \mathbb{R}^d$
- $B^{(0)} = \mathrm{id}$
- $B^{(1)}, \ldots, B^{(m)}$ independent **fractional Brownian motions** with Hurst parameter $H \in (0,1)$, i.e. $B^{(j)}$ zero mean Gaussian process with continuous sample paths and

$$\mathbf{E}|B_t^{(j)} - B_s^{(j)}|^2 = |t-s|^{2H}, \quad s, t \in [0,1]$$

SDEs driven by FBM (cont'd)

$$(SDE) \quad dX_t = \sum_{j=0}^{m} \sigma^{(j)}(X_t)dB_t^{(j)}, \quad t \in [0,1], \qquad X_0 = x_0 \in \mathbb{R}^d$$

Coutin, Qian (2002): Pathwise existence and uniqueness for $H > 1/4$ using rough path theory (Lyons 1994, 1998)

Lin (1995), Klingenhöfer, Zähle (1999); Mikosch, Norvaiša (2000); Ruzmaikina (2000); Nualart, Răşcanu (2002); Errami, Russo (2003); Gubinelli (2004); Nourdin, Simon (2007); Friz, Victoir (2010); ...

$H = 1/2$: classical Stratonovich SDE driven by Brownian motion

Other approach: Fractional Wick-Itô-Skorohod integral

Biagini, Hu, Øksendal, Sulem (2002); Mishura (2003); ...

SDEs driven by FBM (cont'd)

$$\sigma = (\sigma^{(0)}, \sigma^{(1)}, \ldots, \sigma^{(m)}), \quad B = (B^{(0)}, B^{(1)}, \ldots, B^{(m)})$$

$$(SDE) \quad dX_t = \sigma(X_t) dB_t, \quad t \in [0,1], \qquad X_0 = x_0 \in \mathbb{R}^d$$

Pathwise ex. and uniq. for $H > 1/4$, in particular for $H > 1/3$:

$$X = F(x_0, B, \mathbf{B}^2)$$

where

- F locally Lipschitz in appropriate Hölder spaces
- $\mathbf{B}^2$ Lévy area associated to B, i.e.

$$\mathbf{B}^2_{st}(i,j) := \int_s^t (B_u^{(i)} - B_s^{(i)})\, d^\circ B_u^{(j)}, \; i,j = 0, \ldots, m, \; 0 \leq s \leq t \leq 1$$

(symmetric Russo-Vallois integrals)

The Problem

Assumptions:

(i) $\sigma \in C^3(\mathbb{R}^d; (\mathbb{R}^d)^{m+1})$ bounded with bounded derivatives

(ii) B fBm with $H > 1/3$

Problem: Approximation of X on $[0, 1]$

Here: Construction of an approximation Z^n based on

(i) $B_{1/n}, B_{2/n}, \ldots, B_1$,

(ii) x_0 and evaluations of σ and its derivatives,

i.e. Z^n **implementable** numerical scheme

Method of Wood-Chan and Davies-Harte:
Exact simulation of $B_{1/n}, B_{2/n}, \ldots, B_1$ with cost $\mathcal{O}(n \log(n))$

Known Results on Numerical Methods for (SDE)

One-dimensional case ($m = d = 1$)

- $H > 1/2$: Euler scheme Nourdin (2006), N, Nourdin (2007)
- $H > 1/4$: Taylor-type schemes Gradinaru, Nourdin (2009), ...
- $H > 1/2$: Optimal schemes (wrt L^2 error) N (2006, 2008)
- ...

Multi-dimensional case ($m > 1$)

- $H > 1/2$: Euler scheme Mishura, Shevchenko (2008), ...
- Euler scheme for additive noise: Garrido-Atienza, Kloeden, N (2008)
- $H > 1/3$: No implementable and convergent schemes known

Remark Euler scheme does converge to Itô solution and not to X for $H = 1/2$

Modified Milstein Scheme

$$
\begin{aligned}
Z_0^n &= x_0 \\
Z_{k+1}^n &= Z_k^n + \sum_{i=0}^{m} \sigma^{(i)}(Z_k^n)(B^{(i)}_{(k+1)/n} - B^{(i)}_{k/n}) \\
&\quad + \frac{1}{2} \sum_{i,j=0}^{m} \mathcal{D}^{(i)} \sigma^{(j)}(Z_k^n)(B^{(i)}_{(k+1)/n} - B^{(i)}_{k/n})(B^{(j)}_{(k+1)/n} - B^{(j)}_{k/n})
\end{aligned}
$$

where $\mathcal{D}^{(i)} = \sum_{l=1}^{d} \sigma_l^{(i)} \partial_{x_l}$

Extension to $[0,1]$ by piecewise linear interpolation

$$
Z_t^n = Z_k^n + (nt - k)(Z_{k+1}^n - Z_k^n), \qquad t \in [k/n, (k+1)/n)
$$

Then: Convergence of Z^n to X

Milstein Scheme

Davie (2008); Friz, Victoir (2010)

$$\widetilde{Z}^n_{k+1} = \widetilde{Z}^n_k + \sum_{i=0}^{m} \sigma^{(i)}(\widetilde{Z}^n_k)(B^{(i)}_{(k+1)/n} - B^{(i)}_{k/n}) + \sum_{i,j=0}^{m} \mathcal{D}^{(i)}\sigma^{(j)}(\widetilde{Z}^n_k)\mathbf{B}^{\mathbf{2}}_{k/n\,(k+1)/n}(i,j)$$

Then: Convergence of $\widetilde{Z}^n$ to X

But: Law of $\mathbf{B}^2_{st}(i,j) = \int_s^t (B^{(i)}_u - B^{(i)}_s)\, d^\circ B^{(j)}_u$ unknown in general

First construction of Z^n:

Replace $\mathbf{B}^{\mathbf{2}}_{k/n\,(k+1)/n}(i,j)$ by $\frac{1}{2}(B^{(i)}_{(k+1)/n} - B^{(i)}_{k/n})(B^{(j)}_{(k+1)/n} - B^{(j)}_{k/n})$

Second construction of Z^n:

In the following (with convergence proof)

Step 1: Wong-Zakai Approximation

Replace B in (SDE) by

$$B_t^n = B_{k/n} + (nt - k)(B_{(k+1)/n} - B_{k/n}), \qquad t \in [k/n, (k+1)/n)$$

i.e. piecewise linear interpolation of B with stepsize $1/n$

$$(WZ) \qquad Y_t^n = x_0 + \sum_{j=0}^{m} \int_0^t \sigma^{(j)}(Y_s^n) dB_s^{(j),n}, \qquad t \in [0,1]$$

Then: (WZ) ordinary differential equation

$$\dot{Y}_t^n = \sum_{j=0}^{m} \sigma^{(j)}(Y_t^n) \dot{B}_t^{(j),n}, \quad t \in [0,1], \qquad Y_0^n = x_0$$

with $\quad \dot{B}_t^{(j),n} = n(B_{(k+1)/n}^{(j)} - B_{k/n}^{(j)})$ for $t \in (k/n, (k+1)/n)$

Wong-Zakai Approximation

Notation: $\|f\|_{\lambda,\infty} = \sup_{t\in[0,1]} |f(t)| + \sup_{s,t\in[0,1]} \frac{|f(t)-f(s)|}{|t-s|^{\lambda}}$

Theorem I Deya, N, Tindel (2010)

$$\|X - Y^n\|_{\kappa,\infty} \leq \xi_{\sigma,\kappa,H} \cdot \sqrt{\log(n)} \cdot n^{-(H-\kappa)}$$

for all $1/3 < \kappa < H$, where $\xi_{\sigma,\kappa,H}$ positive and finite RV

Remarks

- For σ affine-linear: localisation
- Coutin, Qian (2002): convergence of Y^{2^n} to X in p-variation norm
- Error bound sharp for $dX_t = dB_t$
- (WZ) semidiscretisation, in general not implementable

Theorem I: Proof

Lipschitzness of Itô-Lyons map F:

$$\|X - Y^n\|_{\kappa,\infty} \leq \xi^{(0)}_{\sigma,\kappa,H} \cdot \left(\|B - B^n\|_{\kappa,\infty} + |\mathbf{B^2} - \mathbf{B}^{\mathbf{2},n}|_{2\kappa} \right)$$

for all $1/3 < \kappa < H$ where

$$\mathbf{B}^{\mathbf{2},n}_{st}(i,j) := \int_s^t (B^{(i),n}_u - B^{(i),n}_s)\, dB^{(j),n}_u$$

(Lévy area for B^n) and

$$|f|_{2\kappa} := \sup_{s,t \in [0,1]} \frac{|f(s,t)|}{|t-s|^{2\kappa}}$$

(i) Modulus of continuity of fBm:

$$\|B - B^n\|_{\kappa,\infty} \leq \xi^{(1)}_{\kappa,H} \cdot \sqrt{\log(n)} \cdot n^{-(H-\kappa)}$$

Theorem I: Proof

(i) Modulus of continuity of fBm:

$$\|B - B^n\|_{\kappa,\infty} \leq \xi^{(1)}_{\kappa,H} \cdot \sqrt{\log(n)} \cdot n^{-(H-\kappa)}$$

(ii) $\int_s^t B_u^{(i),n} \, dB_u^{(j),n}$ trapezoidal rule for $\int_s^t B_u^{(i)} \, dB_u^{(j)}$:

$$\left(\mathbf{E}|\mathbf{B}^{\mathbf{2}}_{st} - \mathbf{B}^{\mathbf{2},n}_{st}|^p\right)^{1/p} \leq K_p \cdot |t-s|^{2\kappa+\epsilon} \cdot n^{-2(H-\kappa)+\epsilon}$$

Garcia-Rodemich-Rumsey ineq. and Borell-Cantelli Lemma:

$$|\mathbf{B}^{\mathbf{2}} - \mathbf{B}^{\mathbf{2},n}|_{2\kappa} \leq \xi^{(2)}_{\kappa,\varepsilon,H} \cdot n^{-2(H-\kappa)+\varepsilon}$$

(iii) Lipschitzness of Itô-Lyons map:

$$\|X - Y^n\|_{\kappa,\infty} \leq \xi_{\sigma,\kappa,H} \cdot \sqrt{\log(n)} \cdot n^{-(H-\kappa)}$$

Step 2: Discretising the WZ-Approximation

(WZ) on $(k/n, (k+1)/n)$:

$$\dot{Y}_t^n = n \sum_{j=0}^{m} \sigma^{(j)}(Y_t^n)(B_{(k+1)/n}^{(j)} - B_{k/n}^{(j)})$$

2nd order ODE-Taylor scheme with stepsize $1/n$ applied to (WZ): Modified Milstein scheme

$$\begin{aligned} Z_{k+1}^n = Z_k^n &+ \sum_{i=0}^{m} \sigma^{(i)}(Z_k^n)(B_{(k+1)/n}^{(i)} - B_{k/n}^{(i)}) \\ &+ \frac{1}{2} \sum_{i,j=0}^{m} \mathcal{D}^{(i)}\sigma^{(j)}(Z_k^n)(B_{(k+1)/n}^{(i)} - B_{k/n}^{(i)})(B_{(k+1)/n}^{(j)} - B_{k/n}^{(j)}) \end{aligned}$$

Discretising the WZ-Approximation

Theorem II Deya, N, Tindel (2010)

$$\|X - Z^n\|_{\kappa,\infty} \leq \xi_{\sigma,\kappa,H} \cdot \sqrt{\log(n)} \cdot n^{-(H-\kappa)}$$

for all $1/3 < \kappa < H$

Remarks

- For σ affine-linear: localisation
- Discretisation of (WZ) with 1st order scheme (e.g. Euler): no convergence to X
- Discretisation of (WZ) with arbitrary 2nd order ODE scheme (e.g Heun, Runge-Kutta 4): Theorem II remains valid
- Error bound sharp for $dY_t = dB_t$

Theorem II: Proof

"Similar" to error analysis of ODEs

(i) One step error $\mathcal{O}(\|B\|_{\lambda,\infty} \cdot n^{-3\lambda})$ for all $\lambda < H$

(ii) Error propagation:

$Y^{n;s,a}$ solution of $dY_t = \sigma(Y_t)\, dB_t^n, \ t \geq s, \quad Y_s = a$

Lipschitzness of F:

$$\left| Y^{n;s,a} - Y^{n;s,b} \right|_{\kappa;[s,t]} \leq \xi^{(3)}_{\kappa,\sigma,H} \cdot |a-b|$$

where $|f|_{\kappa;[s,t]} := \sup_{u,v\in[s,t]} \frac{|f(u)-f(v)|}{|u-v|^{\kappa}}$

Thus

$$\sup_{k,l=1,\ldots,n,k\neq l} \frac{|(Z_k^n - Y_{k/n}^n) - (Z_l^n - Y_{l/n}^n)|}{|(k-l)/n|^{\kappa}} \leq \xi^{(4)}_{\sigma,\kappa,\lambda,H} \cdot n^{-3\lambda+1}$$

$+ \ldots +$ Theorem I = Theorem II

Modified Milstein Scheme

$$Z^n_{k+1} = Z^n_k + \sum_{i=0}^{m} \sigma^{(i)}(Z^n_k)(B^{(i)}_{(k+1)/n} - B^{(i)}_{k/n})$$
$$+ \frac{1}{2} \sum_{i,j=0}^{m} \mathcal{D}^{(i)}\sigma^{(j)}(Z^n_k)(B^{(i)}_{(k+1)/n} - B^{(i)}_{k/n})(B^{(j)}_{(k+1)/n} - B^{(j)}_{k/n})$$

Theorem II Deya, N, Tindel (2010)

$$\|X - Z^n\|_{\kappa,\infty} \leq \xi_{\sigma,\kappa,H} \cdot \sqrt{\log(n)} \cdot n^{-(H-\kappa)}$$

for all $1/3 < \kappa < H$

Questions

- Convergence rate in supremum norm, i.e. for $\kappa = 0$?
- Z^n optimal scheme based on $B_{1/n}, \ldots, B_1$?

Numerical example

Test for convergence rates in $\|\cdot\|_\infty$-norm

Conjecture $\quad \|X - Z^n\|_\infty \approx \sqrt{\log(n)}(n^{-H} + n^{-2H+1/2})$

Linear equation

$$\begin{aligned} dX_t^{(1)} &= X_t^{(2)}\, dB_t^{(1)}, \quad t \in [0,1], \qquad X_0^{(1)} = 1 \\ dX_t^{(2)} &= X_t^{(1)}\, dB_t^{(2)}, \quad t \in [0,1], \qquad X_0^{(2)} = 2 \end{aligned}$$

Set

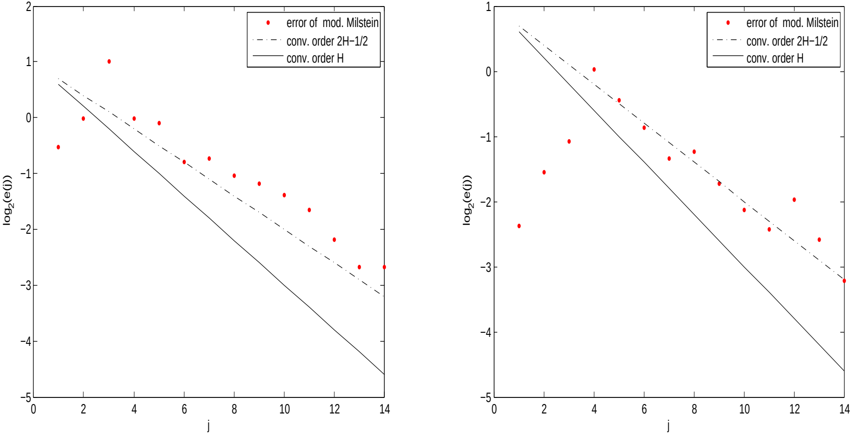
$$e(j) = \sup_{t\in[0,1]} \left| Z_t^{2^j} - Z_t^{2^{j+1}} \right|$$

Then

$$\log_2 e(j) \approx \alpha - \beta \cdot j \quad \text{iff} \quad \sup_{t\in[0,1]} \left| Z_t^{2^j} - X_t \right| \approx (2^j)^{-\beta}$$

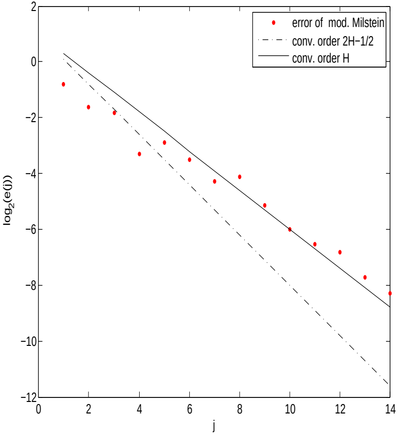
Numerical example I

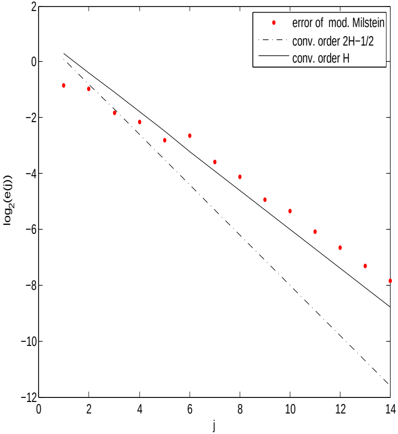
Test equation for $H = 0.4$



Numerical example II

Test equation for $H = 0.7$





Summary

Approximation of SDEs driven by fBm with $H > 1/3$

- Construction of convergent and implementable schemes without Lévy area terms:

  Discretise the Wong-Zakai approximation

- Convergence rate in $\| \cdot \|_{\kappa,\infty}$-norm: $\sqrt{\log(n)}\, n^{-(H-\kappa)}$ $(1/3 < \kappa < H)$

- Convergence rate in $\| \cdot \|_{\infty}$-norm: $\sqrt{\log(n)}(n^{-H} + n^{-2H+1/2})$ (Conjecture!)

Extension to $H > 1/4$ possible:
"only" error bounds for the second iterated itegrals required