A new view on fractional Lévy models

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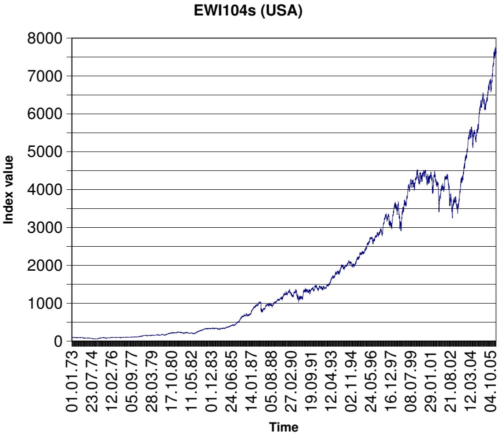
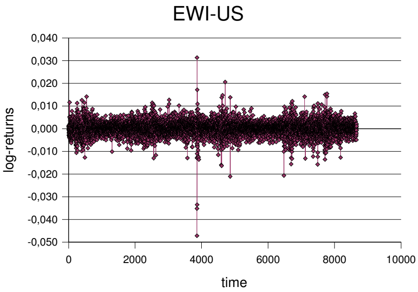
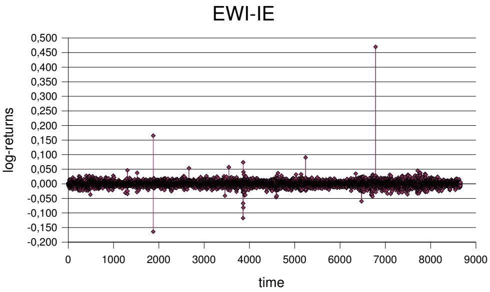
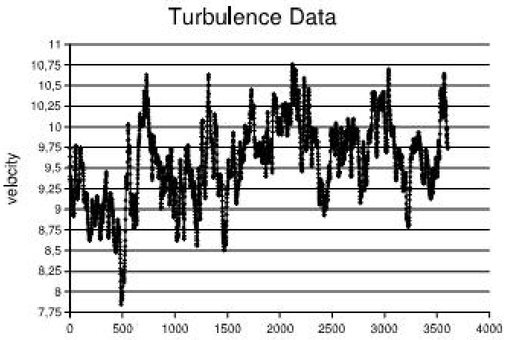
- Motivation for non-semimartingale models
- Fractional Brownian motion models
- Fractional Lévy models

Motivation


EWI104s (USA)
Index value
8000
7000
6000
5000
4000
3000
2000
1000
0
01.01.73
23.07.74
12.02.76
05.09.77
28.03.79
17.10.80
11.05.82
01.12.83
24.06.85
14.01.87
05.08.88
27.02.90
19.09.91
12.04.93
02.11.94
24.05.96
16.12.97
08.07.99
29.01.01
21.08.02
12.03.04
04.10.05
Time


EWI-US
0,040
0,030
0,020
0,010
0,000
-0,010
-0,020
-0,030
-0,040
-0,050
log-returns
0
2000
4000
6000
8000
10000
time


EWI-IE
log-returns
0,500
0,450
0,400
0,350
0,300
0,250
0,200
0,150
0,100
0,050
0,000
-0,050
-0,100
-0,150
-0,200
0
1000
2000
3000
4000
5000
6000
7000
8000
9000
time


Turbulence Data
velocity
11
10,75
10,5
10,25
10
9,75
9,5
9,25
9
8,75
8,5
8,25
8
7,75
0
500
1000
1500
2000
2500
3000
3500
4000

Motivation

discrete data $X_{t_{n,0}}, \cdots, X_{t_{n,n}}$
$t_{n,n} = t$ =fixed, $\Delta_{n,i} \to 0$ as $n \to \infty$
Classical stochastic volatility model

$$X_t = Y_t + \int_0^t \sigma_s dB_s$$

Question:

Do we have to go **beyond semimartingales** in some situations?
How can we cope with **jumps**?

Possible models may be based on **fractional processes** e.g.

$$X_t = Y_t + \int_0^t \sigma_s dB_s^H + Z_t$$

Turbulence

Kolmogorov's 2/3 law:

In the case of an incompressible viscous fluid with large Reynolds number the mean square of the difference of the velocities at two points lying at a distance R (which is neither too large nor too small) is proportional to

$$r^{2H} \text{ with } H = 1/3.$$

Hence a possible model is a fractional Brownian motion based model with $H = 1/3$:

$$X_t = Y_t + \int_0^t \sigma_s dB_s.$$

Fractional Brownian Motion

A fractional Brownian motion (fBm) with **Hurst parameter** $H \in (0,1)$, $B^H = \{B_t^H, t \geq 0\}$ is a zero mean Gaussian process with the covariance function

$$E(B_t^H B_s^H) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}), \quad s,t \geq 0.$$

The fBm is a **self-similar** process, that is, for any constant $a > 0$, the processes
$\{a^{-H} B_{at}^H, t \geq 0\}$ and $\{B_t^H, t \geq 0\}$ have the same distribution.
For $H = \frac{1}{2}$, B^H coincides with the classical Brownian motion. For $H \in (\frac{1}{2}, 1)$ the process possesses **long memory** and for $H \in (0, \frac{1}{2})$ the behaviour is chaotic.

Modelling of jump

X_t **Lévy process** with distribution function P_t.

$$Ee^{iuX_t} = \hat{P}_t(u) = e^{t(i\mu u - \frac{\sigma^2}{2}u^2 + \int (e^{iuz} - 1 - iuh(z))\nu(dz))}$$

Lévy triplet $(\mu, \sigma^2, \nu(dx))$

A measure for the **activity** of the jump component of a Lévy process is the **Blumenthal-Getoor** index β,

$$\beta = \inf\{\delta > 0 : \int (1 \wedge |x|^\delta)\nu(dx) < \infty\}.$$

This index ensures, that for $p > \beta$ the sum of the p-th power of jumps will be finite.

How can we distinguish between semimaringales and non-semimartingales?

Two important characteristics of Brownian semimartingales:

- **Correlation** between neighbouring increments is zero.

$$\sum_{i=1}^{[nt]} (X_{\frac{i+1}{n}} - X_{\frac{i}{n}})(X_{\frac{i}{n}} - X_{\frac{i-1}{n}}) \to 0$$

- **Quadratic variation** is finite

Correlation based method:

Assuming a model of the type

$$X_t = Y_t + \int_0^t \sigma_s dB_s^H$$

we look at the quantity:

$$S_n = \frac{\sum_{i=1}^{[nT]-1}(X_{\frac{i+1}{n}} - X_{\frac{i}{n}})(X_{\frac{i}{n}} - X_{\frac{i-1}{n}})}{\sum_{i=1}^{[nT]}(X_{\frac{i}{n}} - X_{\frac{i-1}{n}})^2} \to C_H = \frac{1}{2}(2^{2H} - 2)$$

confidence interval for Brownian motion based model:

$$\left[-c_\gamma \sqrt{\frac{\sum_{i=1}^{[nt]} |X_{\frac{i}{n}} - X_{\frac{i-1}{n}}|^4}{3\left(\sum_{i=1}^{[nt]} |X_{\frac{i}{n}} - X_{\frac{i-1}{n}}|^2\right)^2}}, c_\gamma \sqrt{\frac{\sum_{i=1}^{[nt]} |X_{\frac{i}{n}} - X_{\frac{i-1}{n}}|^4}{3\left(\sum_{i=1}^{[nt]} |X_{\frac{i}{n}} - X_{\frac{i-1}{n}}|^2\right)^2}}\right],$$

where c_γ denotes the γ-quantile of a $N(0,1)$-distributed random variable.

Problems:

$\sum_{i=1}^{[nT]}(X_{\frac{i}{n}} - X_{\frac{i-1}{n}})^2$ is not robust to jumps.
Replace it by bipower variation

$$n^{-1+2H} \sum_{i=1}^{[nT]-1} |X_{\frac{i}{n}} - X_{\frac{i-1}{n}}||X_{\frac{i+1}{n}} - X_{\frac{i}{n}}| \xrightarrow{p} K_H \int_0^t \sigma_s^2 ds,$$

with $K_H = E(|B_2 - B_1||B_1 - B_0|) = \frac{2}{\pi}(C_H \arcsin(C_H) + \sqrt{1 - C_H^2})$.

Bipower Version:

$$R_n = \frac{\sum_{i=1}^{[nt]-1} (X_{\frac{i+1}{n}} - X_{\frac{i}{n}})(X_{\frac{i}{n}} - X_{\frac{i-1}{n}})}{\sum_{i=1}^{[nt]-1} |X_{\frac{i+1}{n}} - X_{\frac{i}{n}}||X_{\frac{i}{n}} - X_{\frac{i-1}{n}}|} \xrightarrow{p} \frac{C_H}{K_H},$$

where $K_H = E(|B_2 - B_1||B_1 - B_0|) = \frac{2}{\pi}(C_H \arcsin(C_H) + \sqrt{1 - C_H^2})$

$$\left[\frac{C_H}{K_H} - c_\gamma \sqrt{\frac{v^2 \sum_{i=1}^{[nt]-2} |X_{\frac{i+2}{n}} - X_{\frac{i+1}{n}}|^{4/3} |X_{\frac{i+1}{n}} - X_{\frac{i}{n}}|^{4/3} |X_{\frac{i}{n}} - X_{\frac{i-1}{n}}|^{4/3}}{\mu_{4/3}^3 \left(\sum_{i=1}^{[nt]-1} |X_{\frac{i+1}{n}} - X_{\frac{i}{n}}||X_{\frac{i}{n}} - X_{\frac{i-1}{n}}| \right)^2}}, \right.$$

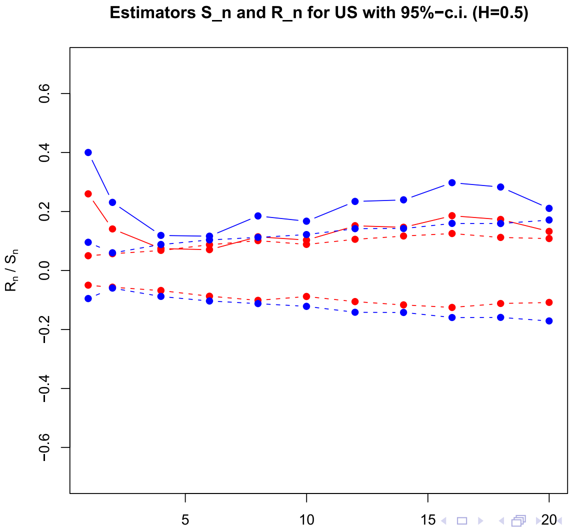
$$\left. \frac{C_H}{K_H} + c_\gamma \sqrt{\frac{v^2 \sum_{i=1}^{[nt]-2} |X_{\frac{i+2}{n}} - X_{\frac{i+1}{n}}|^{4/3} |X_{\frac{i+1}{n}} - X_{\frac{i}{n}}|^{4/3} |X_{\frac{i}{n}} - X_{\frac{i-1}{n}}|^{4/3}}{\mu_{4/3}^3 \left(\sum_{i=1}^{[nt]-1} |X_{\frac{i+1}{n}} - X_{\frac{i}{n}}||X_{\frac{i}{n}} - X_{\frac{i-1}{n}}| \right)^2}} \right].$$

Comparison of S_n and R_n

Inverting S_N and R_n we get estimates for H.
If **no jumps** are present S_n and R_n lead to the **same estimate** for H

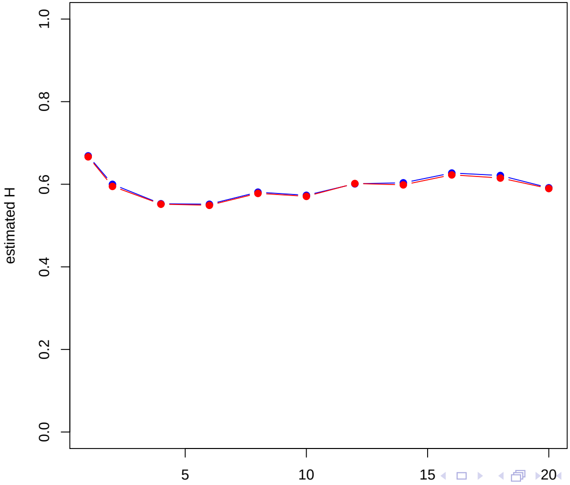
If **jumps are present** R_n leads to the true H, whereas for S_n the denominator gets larger hence H **gets smaller** than the true one.

Hence we can use both to **distinguish between semimartingale and non-semimartingales** and a comparison of both to **detect jumps** even in the non-smimartingale case.

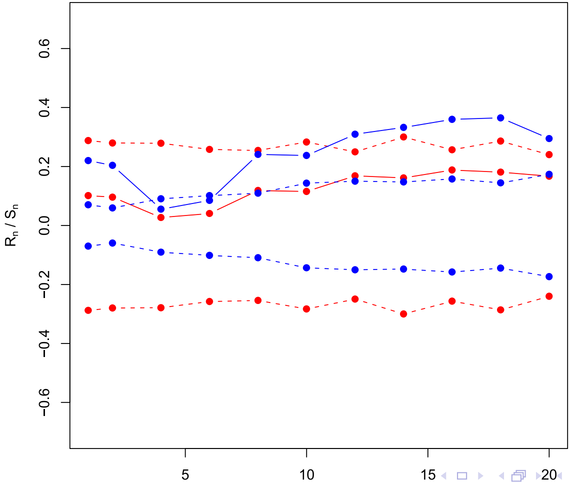


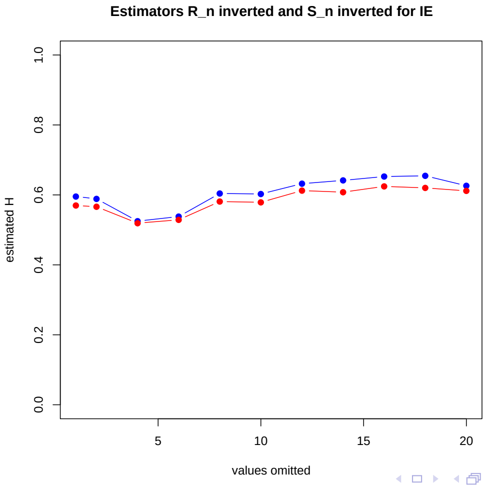
Estimators S_n and R_n for US with 95%−c.i. (H=0.5)
R_n / S_n
0.6
0.4
0.2
0.0
−0.2
−0.4
−0.6
5
10
15
20

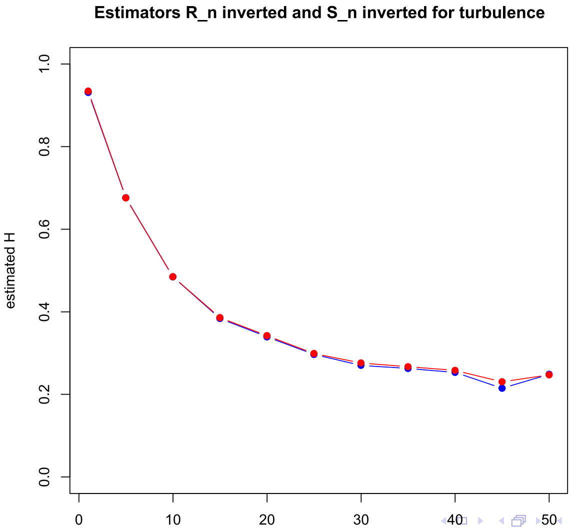
Estimators R_n inverted and S_n inverted for US



Estimators S_n and R_n for IE with 95%−c.i. (H=0.5)




Estimators R_n inverted and S_n inverted for IE
estimated H
values omitted
0.0
0.2
0.4
0.6
0.8
1.0
5
10
15
20


Estimators R_n inverted and S_n inverted for turbulence
estimated H
0.0
0.2
0.4
0.6
0.8
1.0
0
10
20
30
40
50

Comparison of non-normed power variation

$$\sum_i \left| \int_{t_{i-1}}^{t_i} \sigma_s dB_s \right|^p \xrightarrow{p} \begin{cases} 0 & : \quad p > 2 \\ \int_0^t \sigma_s^2 ds & : \quad p = 2 \\ \infty & : \quad p < 2 \end{cases}$$

and the case for the Lévy model

$$\sum_i \left| \int_{t_{i-1}}^{t_i} \sigma_s dL_s \right|^p \xrightarrow{p} \begin{cases} \sum(\left| \int_{u-}^{u} \sigma_s dL_s \right|^p : 0 < u \leq t) & : \quad p > \beta \\ \infty & : \quad p < \beta \end{cases}$$

under appropriate regularity conditions, where β denotes the Blumenthal-Getoor index of L.

Non-normed Power Variation for fractional Brownian motion

$$\sum_i \left| \int_{t_{i-1}}^{t_i} \sigma_s dB_s^H \right|^p \xrightarrow{p} \begin{cases} 0 & : \quad p > 1/H \\ \mu_{1/H} \int_0^t \sigma_s^{1/H} ds & : \quad p = 1/H \\ \infty & : \quad p < 1/H \end{cases},$$

where $H > 1/2$. The integral is a **pathwise Riemann-Stieltjes integral** and we need that σ is a stochastic process with paths of finite q-variation, $q < \frac{1}{1-H}$.
Hence one over the **Hurst exponent** plays a similar role as the **Blumenthal-Getoor index**.

Log-Power Variation Estimators

Theorem

Assume that for some $k \in \mathbb{R}$ and $p \in (a, b)$

$$\left(\frac{1}{n}\right)^{1-pk} V_p^n(X)_t \xrightarrow{p} C, \qquad (1)$$

with a random variable C, $0 < C < \infty$, then

$$\frac{\ln(\frac{1}{n} V_p^n(X)_t)}{p \ln \frac{1}{n}} \xrightarrow{p} \begin{cases} k & : \quad pk \neq 1 \\ 1/p & : \quad pk = 1 \end{cases} \qquad (2)$$

holds as $n \to \infty$.

Models based on a fractional Brownian motion

Theorem

We look at models based on a fractional Brownian motion of the form

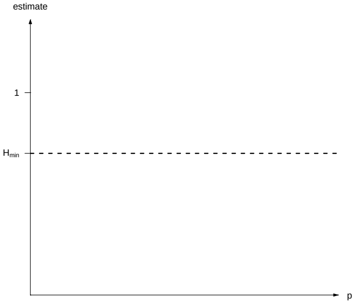
$$X_t = Y_t + \int_0^t \sigma_s dB_s^H + \delta Z_t,$$

where Y Hölder continuous of order $\gamma \in (H, 1]$ and σ possesses strong q variation with $q < \frac{1}{1-H}$ and Z denotes a pure jump process with Blumenthal-Getoor index $\beta < 1/H$, then we obtain

$$\frac{\ln(\frac{1}{n} V_p^n(X)_t)}{p \ln \frac{1}{n}} \xrightarrow{p} \begin{cases} H & : \quad 0 < p \leq 1/H \text{ or } p > 1/H, \quad \delta = 0 \\ 1/p & : \quad p > 1/H, \quad \delta \neq 0 \end{cases}$$

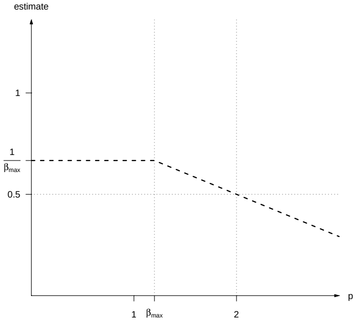
as $n \longrightarrow \infty$.

Purely continuous model



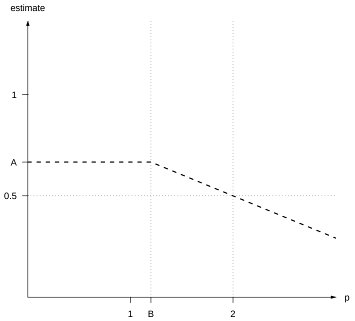
$$0 < H_{min} < 1$$

Pure jump model



$$0 < \beta_{max} < 2, \quad \frac{1}{2} < \frac{1}{\beta_{max}} < \infty$$

Mixed model



$$0 < \beta_{max} < 2, \quad 0 < H_{min} < 1$$
$$0 < \min(\frac{1}{\beta_{max}}, H_{min}) < 1, \quad 1 < \max(\beta_{max}, \frac{1}{H_{min}}) < \infty$$

Normed log-Power Variation

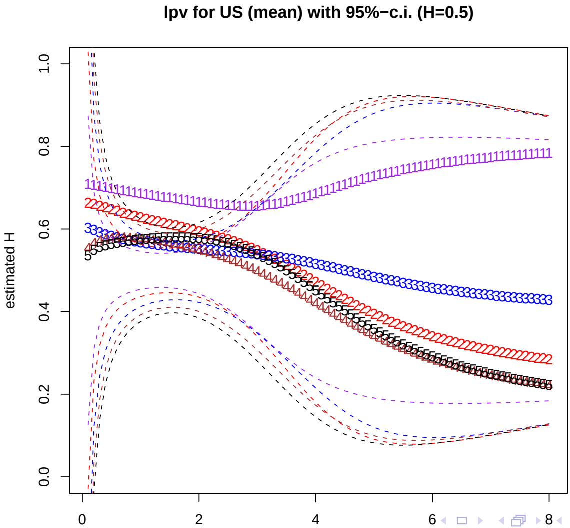
Let

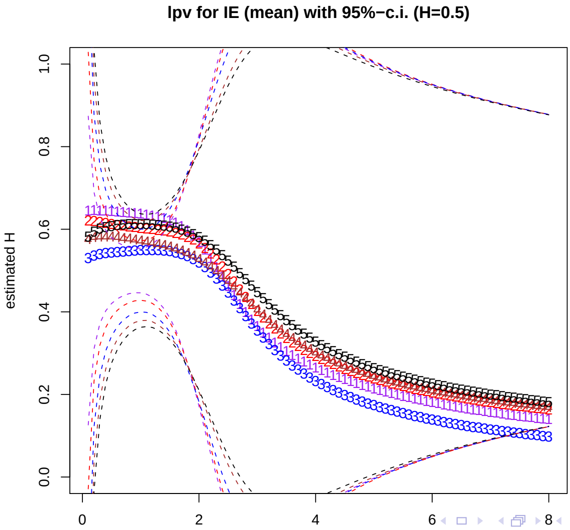
$$X_t = Y_t + \int_0^t \sigma_s dB_s^H + \delta Z_t$$

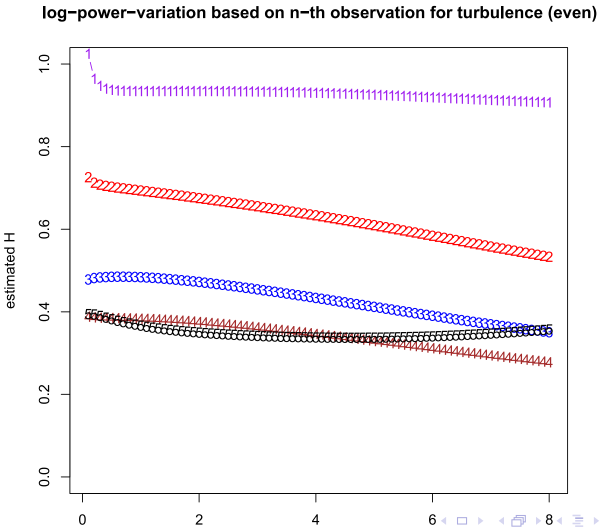
with $H < 3/4$ and assume that $\beta < 1/(2H)$ and Y is Hölder continuous of the order γ with $p(\gamma - H) > 1/2$ and σ is Hölder continuous of the order $a > \frac{1}{2(p \wedge 1)}$, then we obtain for $n \to \infty$ and $p > 0$ if $\delta = 0$ and $\beta/(2(1-\beta H)) < p < 1/(2H)$ if $\delta \neq 0$, then we obtain for $n \to \infty$ and $p > 0$ if $\delta = 0$ and $\beta/(2(1-\beta H)) < p < 1/(2H)$ if $\delta \neq 0$

$$\frac{p \log(2) \sqrt{\mu_{2p}} V_p^n(X)_t}{\sqrt{V_{2p}^n(X)_t (A + B - 2C)}} \left(\frac{\log\left(\frac{V_p^n(X)_t}{2 \sum_{i=1}^{[nt/2]} |X_{\frac{2i}{n}} - X_{\frac{2(i-1)}{n}}|^p} \right)}{p \log(2)} - H \right) \xrightarrow{d} N(0,1),$$

where $\mu_p = \frac{2^{p/2}\Gamma(\frac{p+1}{2})}{\Gamma(1/2)}$, $A = \delta_p(0) + 2\sum_{j\geq 1}(\gamma_p(\rho_H(j)) - \gamma_p(0))$
with $\delta_p(0) = 2^p(\frac{1}{\sqrt{\pi}}\Gamma(p+\frac{1}{2}) - \frac{1}{\pi}\Gamma(\frac{p+1}{2})^2)$,
$\gamma_p(x) = (1-x^2)^{p+\frac{1}{2}} 2^p \sum_{k=0}^{\infty} \frac{(2x)^{2k}}{\pi(2k)!}\Gamma(\frac{p+1}{2}+k)^2$,
$\rho_H(n) = \frac{1}{2}\left((n+1)^{2H} + |n-1|^{2H} - 2n^{2H}\right)$,
$B = 2(\delta_p(0) + 2\sum_{j\geq 1}(\gamma_p(\tilde{\rho}_H(j)) - \gamma_p(0)))$ with
$\tilde{\rho}_H(n) = \frac{\left((2n+2)^{2H} + |2n-2|^{2H} - 2|2n|^{2H}\right)}{2^{2H+1}}$
$C = 2\gamma_p(2^{H-1}) - 2\mu_p^2 + 2\sum_{j\geq 1}(\gamma_p(\bar{\rho}_H(j)) - \gamma_p(0))$ with
$\bar{\rho}_H(n) = \frac{\rho_H(n) + \rho_H(n+1)}{2^H}$


lpv for US (mean) with 95%−c.i. (H=0.5)
estimated H
0.0
0.2
0.4
0.6
0.8
1.0
0
2
4
6
8


lpv for IE (mean) with 95%−c.i. (H=0.5)
estimated H
0.0
0.2
0.4
0.6
0.8
1.0
0
2
4
6
8


log−power−variation based on n−th observation for turbulence (even)
estimated H
0.0
0.2
0.4
0.6
0.8
1.0
0
2
4
6
8

Problems:

- How can we relax the assumption of **normally distributed increments**?

- How can we manage to produce **peaks** like in the energy data?

Fractional Lévy motion

joint with Sebastian Engelke
Idea: Use the same moving average kernel as for fractional Brownian motion.

Linear fractional stable motion (Samorodnitsky and Taqqu (1994))

$$L_{\alpha,H}(t) = \int |t-s|^{H-1/\alpha} - |s|^{H-1/\alpha} S_\alpha(ds),$$

where S_α denotes an α-stable random measure $\alpha \in (0,2)$ and $H \in (0,1)$.
Moving average fractional Lévy motion (Benassi et.al (2004), Marquardt (2006))

$$X_H(t) = \int |t-s|^{H-1/2} - |s|^{H-1/2} L(ds),$$

where L denotes a Lévy process with finite second moment and $H \in (0,1)$.

Moving average fractional Lévy motion

Unifying approach:

$$X_H(t) = \int |t-s|^{H-1/\beta} - |s|^{H-1/\beta} L^\beta(ds),$$

where L denotes a Lévy process with Blumenthal-Getoor index β and finite p-th moment, $p < \beta$ and $H \in (0,1)$.

Properties:

- stationary increments
- for $p < \beta$ the p-th absolute moment exists
- if the p-th absolute moment of the driving Lévy process exists, then it also exists for $H \in (\max(0, 1/\beta - 1/p), 1)$ and is infinite if $H \leq 1/\beta - 1/p$.

Further properties

- Under the additional assumption that L^β is square integrable with $\beta \in (1,2]$ and $H \in (0, 1-(1/\beta - 1/2))$, the we have **long-range dependence** for $H + 1/\beta > 1$ and **chaotic behaviour** for $H + 1/\beta < 1$.

- Under some regularity assumption on the driving Lévy measure we have **infinite strong p-variation** for $p < 1/H$.

Conclusion

- Data of the world index suggests that non-semimartingale models might be suitable for modelling their behaviour.

- Fractional Brownian motion models may explain this behaviour.

- Both correlation based methods and log-power variation may be used to detect jumps in this setting.

- Fractional Lévy motions behave similarly, but additional allow for more flexibility in the distributions and may reproduce peaks as in energy data.