Infinite Divisible Processes: an overview

Jan Rosiński

University of Tennessee

Ambit Processes, Non-Semimartingales and Applications

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Ambit:

1. *circuit, compass*
2. *the bounds or limits of a place or district*
3. *a sphere of action, expression, or influence : scope*

(Merriam-Webster's Online Dictionary)

Addressing Albert Shiryaev's question:

Nualart–Peccati proved (Ann. Probab. 33 (2005)): Let $d \geq 1$ be fixed and suppose that $F_n = I_d(f_n)$ are such that $\mathbb{E}F_n^2 = 1$ and $\lim_{n\to\infty} \mathbb{E}F_n^4 = 3$. Then $F_n \xrightarrow{d} Z$, where $Z \sim N(0,1)$.

During this workshop A. Shiryaev asked the question: *What is special about the homogeneous chaos space that the normal relation between the fourth and the second moment of Z makes Z normal?*

We will show that an intuitive explanation why the limit is Gaussian comes from the structure of independence in the Wiener chaos space.

Proposition (JR and G. Samorodnitsky (1999), Corolaries 5.3–5.4)

Let $F = I_m(f)$ and $G = I_n(g)$ be multiple Itô-Wiener integrals, $m, n \geq 0$. Then

$$\mathrm{Cov}(F^2, G^2) \geq 0.$$

Moreover, F and G are independent if and only if

$$\mathrm{Cov}(F^2, G^2) = 0.$$

Suppose that instead of the limit we have the equality, $\mathbb{E}F_n^4 = 3$, in the Nualart–Peccati Theorem. Let $G_n = I_d(g_n)$ be an independent copy of F_n (this can always be done). Then

$$\mathrm{Cov}((F_n + G_n)^2, (F_n - G_n)^2) = 2\mathbb{E}F_n^4 - 6(\mathbb{E}F_n^2)^2 = 0.$$

By the above Proposition $F_n + G_n \perp\!\!\!\perp F_n - G_n$, and from Bernstein Theorem, $F_n \sim N(0,1)$.
However, since $\mathbb{E}F_n^4 = 3$ only asymptotically, it is natural to expect that $F_n \sim N(0,1)$ also asymptotically. $\square$

Outline

1a. Distributions on Banach spaces

- E separable Banach space, E^* its topological dual;
- the duality action is denoted by $\langle x, y\rangle$, $x \in E$, $y \in E^*$.
- characteristic function of a Borel probability measure μ on E:

$$\widehat{\mu}(y) := \int_E e^{i\langle x,y\rangle}\,\mu(dx), \quad y \in E^*.$$

- μ is infinitely divisible (ID) if $\forall n \in \mathbb{N}\ \exists$ pr. m. μ_n such that $\mu = (\mu_n)^{*\,n}$
- continuous truncation:

$$[\![x]\!] = \begin{cases} x & \text{if } \|x\| \le 1, \\ \frac{x}{\|x\|} & \text{if } \|x\| > 1. \end{cases}$$

Theorem (Lévy-Khintchine representation: Dettweiler (1976))

A probability measure μ on E is ID $\iff$ $\exists$ unique triplet (Σ, ν, b) such that the cumulant of μ is given by

$$\log \widehat{\mu}(y) = -\frac{1}{2}\langle \Sigma y, y\rangle + \int_E \left(e^{i\langle x,y\rangle} - 1 - i\langle [\![x]\!], y\rangle\right) \nu(dx) + i\langle b, y\rangle$$

where $\Sigma : E^* \mapsto E$ is a nonnegative symmetric operator, ν is a Borel measure on E such that $\int \langle x, y\rangle^2 \wedge 1\, \nu(dx) < \infty$ for each $y \in E^*$ and $\mu(\{0\}) = 0$, and $b \in E$.

We write $\mu = ID(\Sigma, \nu, b)$ and call Σ the covariance operator of the Gaussian part of μ, ν the the Lévy measure of μ, and b a shift.

Remarks

- For a general Banach space, such as $E = C[0,1]$, there is no integrability condition characterizing Lévy measures. In particular, condition

$$\int_E \|x\|^2 \wedge 1 \, \nu(dx) < \infty \tag{1}$$

is neither necessary nor sufficient. Also, there is no operator theory condition characterizing covariance operators on a general Banach space.

Remarks (continue)

- (1) is sufficient for Lévy measures when E is a Banach space of type 2 and necessary when E is of cotype 2. The notions of type and cotype for Banach spaces were introduced by J. Hoffmann–Jørgensen.
- If $E = H$ is a Hilbert space then a measure ν on H with $\nu(\{0\}) = 0$ is a Lévy measure if and only if it satisfies (1). A symmetric nonnegative operator $\Sigma : H \mapsto H$ is a covariance operator if and only if it has summable trace.

1b. Distributions on cones of Banach spaces

ID distributions on cones lead to rich classes of subordinators in Banach spaces.

A cone K in E is a closed non-empty set closed under addition and multiplication by nonnegative reals. A cone K induces a partial order on E by defining $x \leq_K y$ whenever $y - x \in K$.

A cone K in E is normal if for every $z \in K$ the set $[0, z] := \{x \in K : x \leq_K z\}$ is bounded.

The normality of a cone is a natural assumption precluding pathological situations.

Let μ be an ID distribution on E concentrated on a normal cone K.

Definition

We say that μ admits the special Lévy-Khintchine representation if

$$\log \widehat{\mu}(y) = \int_K \left(e^{i\langle x,y \rangle} - 1 \right) \nu(dx) + i\langle b_0, y\rangle,$$

where $b_0 \in K$ is called a drift and Lévy measure ν concentrated on the cone K satisfies $\int |\langle x, y\rangle| \wedge 1 \, \nu(dx) < \infty$ for each $y \in E^*$.

Skorohod showed that any ID distribution μ concentrated on a normal cone in R^d has the special Lévy-Khintchine representation.

Is this fact valid in Banach spaces? The answer depends on the type of cone.

A cone K is said to be regular if every K-increasing and K-majorized sequence in K is convergent. That is, for any sequence $(x_n) \subset K$ and $x \in K$ such that $x_n \leq_K x_{n+1} \leq_K x \ \forall n$, $x_\infty := \lim_n x_n$ exists.

In a finite dimensional vector space every proper cone is normal and regular.

Theorem (V. Perez-Abreu, J.R. (2007))

Let K be a normal cone in a separable Banach space E. TFAE:

- *Every ID distribution concentrated on K has special the Lévy-Khintchine representation;*
- *Cone K is regular;*
- *K does not contain an isomorphic copy of the cone c^+ of nonnegative convergent sequences. That is, there is no isomorphic mapping V of c into E such that $Vc^+ \subset K$.*

This also solves a problem posed by E. Dettweiler (1976).

The existence of special representations characterizes cones, not Banach spaces. For example, c_0 contains both regular and not regular cones.

A function $g : E \mapsto \mathbb{R}_+$ is said to be log-subadditive if

$$g(x+y) \leq Kg(x)g(y) \quad \forall x, y \in E.$$

g is locally bounded when $\sup_{\|x\| \leq r} g(x) < \infty$, $r > 0$.

Theorem

Let $g : E \mapsto \mathbb{R}_+$ be a log-subadditive locally bounded function, and let X be an ID random variable in E with Lévy measure ν. Then $\mathbb{E}g(X) < \infty$ if and only if $\int_{\{\|x\|>1\}} g(x)\,\nu(dx) < \infty$.

For example, $g(x) = \|x\|^p$ $(p > 0)$, $g(x) = \exp(\|x\|^\beta)$ $(\beta \in (0, 1])$ are log-subadditive. But $g(x) = \exp(\|x\| \log^+ \|x\|)$ is not.
The case $E = \mathbb{R}^d$ can be found in Sato's book.

Theorem (JR (1995))

Let $X \sim (0, \nu, b)$ *be a random variable in a Banach space* E *with Lévy measure* ν *of bounded support. Assume that* $\nu \neq 0$ *and let*

$$R := \inf\{r > 0 : \nu\{x : \|x\| > r\} = 0\}$$

and

$$p := \nu\{x : \|x\| = R\}.$$

Then

$$\mathbb{E} \exp\left\{R^{-1}\|X\| \log^{+}(\alpha\|X\|)\right\} < \infty$$

for every $\alpha \in (0, \frac{1}{epR})$. *(*$1/0 = \infty$*.)*

Methods: isoperimetric inequalities (see Talangrand (1989)), or hypercontractivity (see Kwapień-Szulga (1991)), combined with a technique similar to *découpage de Lévy* combined with certain methods of de Acosta.

Corollary (Large Deviations, JR (1995))

Let X be an ID random variable in E such that Lévy measure has bounded support and let R be as above. Then

$$\lim_{t \to \infty} \frac{\mathbb{P}\{\|X\| > t\}}{t \log t} = -R^{-1}.$$

Theorem (Houdré (2002))

Let $X \sim ID(0, \nu, b)$ be a random variable in E with Lévy measure ν of bounded support. Let R be as above and $V^2 = \int_E \|x\|^2 \nu(dx)$ Then for every Lipschitz function $f : E \mapsto \mathbb{R}$ with $\|f\|_{Lip} \leq 1$ and $t > 0$

$$\mathbb{P}(f(X) - \mathbb{E}f(X) \geq t) \leq \exp\left\{\frac{t}{R} - \left(\frac{t}{R} + \frac{V^2}{R^2}\right)\log\left(1 + \frac{Rt}{V^2}\right)\right\}$$

Corollary

Under above notation, for every $\theta < R^{-1}$

$$\mathbb{E}e^{\theta|f(X)|\log^+|f(X)|} < \infty.$$

COVARIANCE REPRESENTATION:

Theorem (C. Houdré, V. Pérez-Abreu, D. Surgailis (1998))

Let $X \sim ID(0, \nu, b)$ be a random vector in $\mathbb{R}^d$ such that $\mathbb{E}\|X\|^2 < \infty$. Let $f, g : \mathbb{R}^d \mapsto \mathbb{R}$ be Lipschitz. Then

$$\begin{aligned} &\mathrm{Cov}(f(X), g(X)) \\ &= \int_0^1 \mathbb{E}_s \int_{\mathbb{R}^d} (f(Y+x) - f(Y))(g(Z+x) - g(Z))\, \nu(dx) ds \end{aligned}$$

where the expectation is with respect to probability measure $\mathbb{P}_s$ on $\mathbb{R}^{2d}$ such that $(Y, Z) \sim ID(0, \nu_s, b_s)$, where $b_s = (b, b)$ and $\nu_s = s\nu_1 + (1-s)\nu_0$, $s \in [0,1]$. Here $\nu_0(du, dv) = \nu(du)\delta_0(dv) + \delta_0(du)\nu(dv)$ is concentrated on the two main 'axes' of $\mathbb{R}^{2d}$ and $\mu_1(du, dv)$ is the push-forward of ν to the main diagonal of R^{2d}, $(u, u) \in \mathbb{R}^{2d}$.

Notice that

$\forall s \in [0,1]$, under $\mathbb{P}_s$, Y and Z have the same distribution as X

$Y \perp\!\!\!\perp Z$ under $\mathbb{P}_0$, and

$Y = Z$ under $\mathbb{P}_1$.

Application of the covariance representation for large deviation estimates

Let $f : \mathbb{R}^d \mapsto \mathbb{R}$ be Lipschitz with $\|f\|_{Lip} \leq 1$, and let $\mathbb{E}e^{t\|X\|} < \infty$, $t \in (0, t_0)$. Let $g(x) = e^{tf(x)}$. Assume for a moment that f is bounded, so that g is also Lipschitz, and that $\mathbb{E}f(X) = 0$.

Consider the Laplace transform $L(t)$ of $f(X)$, $L(t) = \mathbb{E}e^{tf(X)}$.

$$\frac{d}{dt}L(t) = \mathbb{E}f(X)e^{tf(X)} = \mathrm{Cov}(f(X), g(X))$$

By the covariance representation

$$\begin{aligned}
\frac{d}{dt}L(t) &= \int_0^1 \mathbb{E}_s \int_{\mathbb{R}^d} (f(Y+x)-f(Y))(e^{tf(Z+x)}-e^{tf(Z)})\,\nu(dx)ds \\
&\le \int_0^1 \mathbb{E}_s e^{tf(Z)} \int_{\mathbb{R}^d} |f(Y+x)-f(Y)|(e^{t|f(Z+x)-f(Z)|}-1)\,\nu(dx)ds \\
&\le \int_0^1 \mathbb{E}_s e^{tf(Z)} \int_{\mathbb{R}^d} \|x\|(e^{t\|x\|}-1)\,\nu(dx)ds = L(t)h(t),
\end{aligned}$$

where

$$h(t) := \int_{\mathbb{R}^d} \|x\|(e^{t\|x\|}-1)\,\nu(dx).$$

Thus $\dfrac{L'(t)}{L(t)} \le h(t)$, which yields

$$\mathbb{E}e^{tf(X)} = L(t) \le \exp\Big(\int_0^t h(s)\,ds\Big), \quad t \in (0, t_0).$$

By the standard Crameér method in large deviations

$$\mathbb{P}(f(X) \geq a) \leq \exp(-\int_0^a h^{-1}(s)\, ds)$$

Now once can remove restrictions on f to get

$$\mathbb{P}(f(X) - \mathbb{E}f(X) \geq t) \leq \exp(-\int_0^t h^{-1}(s)\, ds).$$

If ν has bounded support, we can bound h easily to get the tail bound given on a previous slide (for finite dim spaces and extend to Banach spaces). It can be applied to other ID random vectors as well (see Houdré (2002)).

For current results on integrability of seminorms of chaos variables see Andreas Basse (2009).

The first moment estimate:

Theorem (M.B. Marcus, JR (2001))

Let $X \sim ID(0, \nu, b)$ be a mean zero random vector in a separable Hilbert space E. Let $\ell = \ell(\nu)$ be a unique solution of the equation

$$\int_E \|\ell^{-1}x\|^2 \wedge \|\ell^{-1}x\| \, \nu(dx) = 1.$$

Then

$$(0.25)\, \ell(\nu) \leq \mathbb{E}\|X\| \leq (2.125)\, \ell(\nu).$$

If ν is symmetric, the the upper bound constant can be decreased to 1.25.

3. Cylindrical random variables and measures - ID and radonification

A cylindrical random variable Y in E is a continuous linear map

$$Y : E^* \mapsto L^0(\Omega, \mathcal{F}, \mathbb{P}).$$

Distribution of Y is a cylindrical measure, i.e., a finitely additive probability measure μ defined on the algebra of cylindrical sets

$$\mu\left(C(y_1, \ldots, y_n; B)\right) := \mathbb{P}\left((Y(y_1), \ldots, Y(y_n)) \in B\right),$$

where $y_1, \ldots, y_n \in E^*$, $B \in \mathcal{B}(\mathbb{R}^n)$, $n \geq 1$, and

$$C(y_1, \ldots, y_n; B) := \{x \in E : (\langle x, y_1 \rangle, \ldots, \langle x, y_n \rangle) \in B\}.$$

A cylindrical measure may become a true measure when pushed forward to some larger space. If $V : E \mapsto F$ is a continuous linear operator into some separable Banach space F and μ is a cylindrical measure on E, then push forward cylindrical measure $\mu \circ V^{-1}$ is defined by

$$\mu \circ V^{-1}(C(w_1, \dots, w_n; B)) := \mu(C(V^* w_1, \dots, V^* w_n; B)),$$

for any $w_1, \dots, w_n \in F^*$, $B \in \mathcal{B}(\mathbb{R}^n)$, where $V^* : F^* \mapsto E^*$ is the dual operator.

The operator V is said to be radonifying μ when $\mu \circ V^{-1}$ has a unique extension to a probability measure on F.

In terms of random variables, if Y is a cylindrical random variable with distribution μ, then $V : E \mapsto F$ radonifies μ if and only if there exists a true random variable $X : \Omega \mapsto F$ such that $\forall w \in F^*$

$$Y(V^* w) = \langle X, w \rangle \quad a.s.$$

The operator V is said to be p-radonifying ($p \in [0, \infty)$) when it radonifies any cylindrical probability measure μ with weak p-th moment:

$$\int_E |\langle x, y \rangle|^p < \infty, \quad \forall y \in E^*.$$

A celebrated result of S. Kwapień and L. Schwartz show that an operator is p-radonifying for some $p \in (1, \infty)$ if and only if it is p-summing.

Convolution of cylindrical measures is well defined.
A cylindrical probability measure μ is said to be ID if $\forall n \in \mathbb{N}\ \exists$ a cylindrical measure μ_n such that $\mu = (\mu_n)^{*\,n}$.

Corollary

Let μ be a cylindrical measure on E that is the distribution of a cylindrical random variable Y. Then μ is ID if and only if all finite dimensional distributions of a stochastic process $\{Y(y) : y \in E^*\}$ are ID.

Every cylindrical ID measure has a cylindrical Lévy-Khintchine representation involving a cylindrical Lévy measure.

—

Badrikian (1970), Dettweiler (1976), ...

One can also define cylindrical Lévy process and consider the related radonification problems.

See, M. Riedle & D. Applebaum (2009), M. Riedle and O. van Gaans (2009).

Such problems naturally arise in SPDE's; cf. Peszat - Zabczyk (2007, book), van Neerven, J.M.A.M., Veraar, M. C. and Weis, L. (2007).

In this context, a cylindrical semimartingale $Y = \{Y_t\}_{t \geq 0}$ on E^* is a continuous linear map

$$Y : E^* \mapsto \mathbb{S}$$

where $\mathbb{S}$ denotes the space of real semimartingales on a given filtered probability space, endowed with the Emery topology.

Theorem (A. Jakubowski, S. Kwapień, P.R. de Fitte, JR (2002))

Let $Y = \{Y_t\}_{t \geq 0}$ be a cylindrical semimartingale on E^*, the dual to a Banach space E. Suppose that an operator $V : E \mapsto F$ can be factored as $V = V_1 V_2$, where $V_1 : E \mapsto G$ and $V_2 : G \mapsto F$ are 2-radonifying operators for some Banach space G. Then V radonifies Y; that is, there exists an F-valued semimaringale $X = \{X_t\}_{t \geq 0}$, such that for every $w \in F^*$ the real processes

$$Y_t(V^* w) = \langle X_t, w \rangle, \quad t \geq 0$$

are indistinguishable.

If E and F are Hilbert spaces, then the conclusion holds when V is only a Hilbert-Schmidt operator.

Remark: This solves the 3-operators problem in Hilbert spaces.

Following an analogy with Gaussian processes we define

Definition

Let T be an arbitrary nonempty set. A process $\mathbf{X} = \{X_t\}_{t \in T}$ is said to be an infinitely divisible (ID) stochastic process if for any $t_1, \ldots, t_n \in T$ the random vector

$$(X_{t_1}, \ldots, X_{t_n})$$

has an ID distribution.

In the fundamental work Maruyama (1970) defines a cylindrical Lévy measure of an ID process on $\mathbb{R}^T$. Then he extended this finite additive measure of a countably additive on a special σ-ring of subsets of $\mathbb{R}^T$. Such σ-ring has a complicated structure when the index set T is uncountable.

However, there is a problem with Maruyama's proof (pointed to me by Adam Jakubowski, private communication).

Notation:

$$\mathbb{R}^T = \{x : T \mapsto \mathbb{R}\}$$

The cylindrical σ-field

$$\mathcal{B}^T = \prod_{t \in T} \mathcal{B}(R_t) \qquad (R_t = \mathbb{R})$$

$$\mathbf{O}_S := \{x \in \mathbb{R}^T : x_t = 0 \ \forall t \in S\}$$

Notice that $\mathbf{O}_T \notin \mathcal{B}^T$, when T is uncountable.

How to define a Lévy measure on $\mathbb{R}^T$ that would guarantee its uniqueness?

Definition

A measure ν on the cylindrical σ-field $\mathcal{B}^T$ is said to be a Lévy measure on $\mathbb{R}^T$ if the following two conditions hold

(i) for every $t \in T$

$$\int_{\mathbb{R}^T} \left(|x_t|^2 \wedge 1\right) \nu(dx) < \infty$$

(ii) for every $A \in \mathcal{B}^T$ there exists a countable set $T_A \subset T$ such that

$$\nu(A) = \nu(A \setminus \mathbf{0}_{T_A}).$$

Remark

If T is a countable index set, then condition (ii) is equivalent to

$$\nu\{\mathbf{O}_T\} = 0.$$

Proof: Assume T is countable. If (ii) holds then for $A = \mathbf{O}_T$

$$\nu(\mathbf{O}_T) = \nu(\mathbf{O}_T \setminus \mathbf{O}_{T_A}) \leq \nu(\mathbf{O}_T \setminus \mathbf{O}_T) = 0.$$

If $\nu(\mathbf{O}_T) = 0$ then we always take $T_A = T$. $\square$

Remark

Suppose that condition (i) holds and

(iii) $\exists$ a countable set $T_0 \subset T$ such that

$$\nu\{\mathbf{O}_{T_0}\} = 0.$$

Then ν is a Lévy measure.

Proposition

A Lévy measure is σ-finite if and only if condition (iii) holds.

Theorem (Lévy-Khintchine representation)

Let $\mathbf{X} = \{X_t\}_{t \in T}$ be an infinitely divisible stochastic process. Then there exist a unique generating triplet (Σ, ν, b) consisting of

(i) *a nonnegative symmetric operator $\Sigma : \mathbb{R}^{(T)} \mapsto \mathbb{R}^T$,*

(ii) *a Lévy measure ν on $\mathbb{R}^T$,*

(iii) *a function $b \in \mathbb{R}^T$,*

such that for any $y \in \mathbb{R}^{(T)}$

$$\log \mathbb{E} \exp(i \sum_{t \in T} y_t X_t) =$$

$$-\frac{1}{2}\langle y, \Sigma y \rangle + \int_{\mathbb{R}^T} \left(e^{i\langle y, x \rangle} - 1 - i\langle y, [\![x]\!]\rangle \right) \nu(dx) + i\langle y, b \rangle.$$

Here $\langle y, x \rangle = \sum_{t \in T} y_t x_t$, $[\![x]\!]_t = x_t/(|x_t| \vee 1)$, $x \in \mathbb{R}^T$, $y \in \mathbb{R}^{(T)}$; $\mathbb{R}^{(T)}$ is the set of functions from T to $\mathbb{R}$ with finite support.

Remarks about the proof:

For any finite set $\Gamma \subset T$, $X_\Gamma = \{X_t\}_{t\in\Gamma}$ is an ID random vector in finite dimensional space $\mathbb{R}^\Gamma$. Thus it has the generating triplet $(\Sigma_\Gamma, \nu_\Gamma, b_\Gamma)$. The families

$$\{\Sigma_\Gamma : \text{finite } \Gamma \subset T\} \quad \text{and} \quad \{b_\Gamma : \text{finite } \Gamma \subset T\}$$

are consistent and produce immediately $\Sigma : \mathbb{R}^{(T)} \mapsto \mathbb{R}^T$ and $b \in \mathbb{R}^T$, respectively. However,

$$\{\nu_\Gamma : \text{finite } \Gamma \subset T\}$$

does not constitute a projective system of measures in the usual sense. For example, if $T = \{1,2\}$, $\nu_{\{1,2\}} = \delta_{(0,1)}$, then $\nu_{\{1\}} = 0$ is not a projection of $\delta_{(0,1)}$ onto the first axis. Thus results on limits of projective systems of measures do not apply, which is the main difficulty.

EXAMPLES OF LÉVY MEASURES FOR PROCESSES:

1. Lévy processes.

$\mathbf{X} = \{X_t\}_{t \geq 0}$ be a Lévy process with

$$\mathbb{E} e^{iuX_t} = e^{t\psi(u)},$$

$$\psi(u) = \int_{-\infty}^{\infty} (e^{iuv} - 1 - iu[\![v]\!])\,\eta(dv).$$

Here $T = \mathbb{R}_+$. What is the Lévy measure ν of $\mathbf{X}$ on the path space $\mathbb{R}^{\mathbb{R}_+}$?

ANSWER: Path Lévy measure ν of a Lévy process $\mathbf{X}$ is the push-forward measure of $\eta \times \mathrm{Leb}$ by the map

$$\mathbb{R} \times \mathbb{R}_+ \ni (v, s) \mapsto v\mathbf{1}_{[s,\infty)} \in \mathbb{R}^{\mathbb{R}_+}.$$

Therefore, path Lévy measure of a Lévy process is concentrated on the set of one-step functions

$$S := \{v\mathbf{1}_{[s,\infty)} : v \in \mathbb{R},\ s \geq 0\}.$$

(Precisely, $\nu_*(\mathbb{R}^{\mathbb{R}_+} \setminus S) = 0$.)

Actually, an ID process $\mathbf{X} = \{X_t\}_{t\geq 0}$ with the generating triplet $(0, \nu, b)$ has independent increments if and only if the support of ν is included in S.

This indicates the richness of the class of ID processes.

For a Poisson process with parameter λ,

$$\text{supp}\ \nu = \{\mathbf{1}_{[s,\infty)} : s \geq 0\}$$

and ν is the image measure of $(\lambda\delta_1) \times \text{Leb}$ by the map

$$s \mapsto \mathbf{1}_{[s,\infty)}$$

Bad properties such as discontinuities, non differentiability, etc, of sample paths of an ID process are inherited from the corresponding properties of the support of its Lévy measure.

For a precise statement see JR (1989).

2. ID point processes.

Let N be an ID point process on $\mathbb{R}^d$. Thus,

$$\{N(A) : A \in \mathcal{B}_0(\mathbb{R}^d)\}$$

is an ID process indexed by bounded Borel subsets of R^d. Its Lévy measure ν is obtained on the cylindrical σ-field of $R^{\mathcal{B}_0(\mathbb{R}^d)}$.

It can be shown that ν is concentrated on $\mathbf{N}_{\mathbb{R}^d}$, the space of nonnegative integer-valued measures, finite on bounded Borel sets.

The restriction of ν to $\mathbf{N}_{\mathbb{R}^d}$ is known as KLM measure of N.

Benefits of having generating triplets on path spaces:

- possibility of constructing various series representations of processes which are crucial in the study of sample path properties
- availability of stochastic integral representations resulting from various representations of the path Lévy measure; e.g., Itô-Lévy representation of any ID process
- unified approach, parametrization

Definition

Let $T = \mathbb{Z}^n$ or $\mathbb{R}^n$. A stochastic processes $\{X_t\}_{t\in T}$ is said to be stationary if $\forall\ h \in T$

$$\{X_{t+h}\}_{t\in T} \stackrel{d}{=} \{X_t\}_{t\in T}.$$

If $\{X_t\}_{t\in T}$ is ID with generating triplet $(0, \nu, b)$, then stationarity of the process is equivalent that ν is invariant under the shift and b is a constant on T.

Here $T = \mathbb{Z}$ or $\mathbb{R}$.

Theorem (JR and T. Żak (1996))

Let $\{X_t\}_{t\in T}$ be a stationary ID process such that the Lévy measure of X_0 has no atoms in $2\pi\mathbb{Z}$. Then $\{X_t\}_{t\in T}$ is mixing if and only if

$$\lim_{t\to\infty} \mathbb{E}e^{i(X_t - X_0)} = |\mathbb{E}e^{iX_0}|^2.$$

Codifference: $\rho(t) = \log \mathbb{E} e^{i(X_t - X_0)} - \log \mathbb{E} e^{iX_t} - \log \mathbb{E} e^{-iX_0}$.
It it the covariance function for Gaussian processes.

Corollary

Under the assumptions of the previous theorem, $\{X_t\}_{t \in T}$ is mixing if and only if $\rho(t) \to 0$ as $t \to \infty$.

Theorem

Let $\{X_t\}_{t \in T}$ be a stationary ID process. Then the process is ergodic if and only if it is weakly mixing.

Corollary

$\{X_t\}_{t \in T}$ is ergodic (equivalently, weakly mixing) if and only if

$$\lim_{t \to \infty} t^{-1} \int_0^t |\rho(s)| \, ds = 0,$$

where ρ is the codifference function of the process.

6.a. Stationary symmetric stable processes

Theorem (JR (1995))

$\{X_n\}_{n\in\mathbb{Z}}$ is a stationary $S\alpha S$ process if and only if there exist an $S\alpha S$ noise Λ on a Borel subset S of $\mathbb{R}^d$ equipped with a σ-finite control measure λ such that for every $n \in \mathbb{Z}$

$$X_n = \int_S a_n(s) \left(\frac{d\lambda \circ \phi^n}{d\lambda}(s) \right)^{1/\alpha} f \circ \phi^n(s)\,\Lambda(ds) \quad a.s.$$

Here $\phi : S \mapsto S$ is a nonsingular bijection, $\{a_n\}$ is a sequence of $\{-1,1\}$-valued functions satisfying

$$a_{n+1}(s) = a_n(\phi(s))a_1(s), \quad n \in \mathbb{Z},\ s \in S,$$

and $f \in L^\alpha(S,\lambda)$.

(For the case $T = Z^d$ or $\mathbb{R}^d$ see JR (2000)).

By the recurrence structure we get:

$$a_n = \prod_{j=0}^{n-1} u \circ \phi^j, \quad n \in \mathbb{Z}_+$$

where $u = a_1$ is a $\{-1, 1\}$-valued measurable function on S and

$$\frac{d\lambda \circ \phi^n}{d\lambda} = \prod_{j=0}^{n-1} v \circ \phi^j, \quad n \in \mathbb{Z}_+$$

where $v = \frac{d\lambda \circ \phi}{d\lambda}$.
Therefore, a stationary SαS process is determined by (ϕ, f, u). The most important is ϕ.

Long run behavior of the deterministic flow

$$\{\phi^n\}_{n\in\mathbb{Z}}$$

determines long range dependence of the process $\{X_n\}$.

Decompositions of the flow introduce decompositions of the stationary stable process into independent parts with different dependence structures and different ergodic behavior.

For example, Hopf decomposition of the flow into conservative and dissipative parts distinguishes long range dependent, generally non ergodic part of a stable process and an independent, generally short range dependent mixing part. See JR (1995) for processes, and (2000) for random fields. Extensions of these ideas to stable processes and fields were obtained by S. Kolodynski and JR (2002), Pipiras and Taqqu (2002), JR and G. Samorodnitsky (1996), G. Samorodnitsky (2004), (2005), P. Roy and G. Samorodnitsky (2008), P. Roy (2010), and more.

6.b Stationary seldecomposable processes

Any stationary mean-zero selfdecomposable process $\{X_n\}_{n\in\mathbb{Z}}$ can be represented as

$$X_n = \int_0^\infty g(V^n(t))\,dZ_t, \quad n \in \mathbb{Z}$$

where $V : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a Lebesgue measure preserving transformation of $\mathbb{R}_+$ and $\{Z_t\}_{t\geq 0}$ is a Lévy process with

$$\mathbb{E}e^{iuX_t} = e^{t\psi(u)},$$

where $\psi(u) = \int_0^1 (e^{iux} - 1 - iux)x^{-1}\,dx$.
Thus $\{X_n\}$ can be viewed as a process in the first order chaos of $\{Z_t\}_{t\geq 0}$.

Theorem

Every stationary zero mean selfdecomposable process $\{X_n\}_{n\in\mathbb{Z}}$ *can be written uniquely in distribution as the sum*

$$X_n = \sum_{i=0}^{4} X_n^{(i)}, \quad n \in \mathbb{Z},$$

where $\{X_n^{(i)}\}_{n\in\mathbb{Z}}$, $i = 0, \ldots, 4$ *are independent stationary zero mean selfdecomposable process (some may be zero) such that*

(0) $\{X_n^{(0)}\}_{n\in\mathbb{Z}}$ *has constant paths;*

(1) $\{X_n^{(1)}\}_{n\in\mathbb{Z}}$ *is not ergodic;*

(2) $\{X_n^{(2)}\}_{n\in\mathbb{Z}}$ *is weakly mixing (and so ergodic) but not mixing;*

Theorem (continue)

(3) $\{X_n^{(3)}\}_{n\in\mathbb{Z}}$ *is mixing and does not have mixed moving average component;*

(4) $\{X_n^{(4)}\}_{n\in\mathbb{Z}}$ *is a mixed moving average process.*

D. Nualart and Schoutens (2000) gave a chaotic decomposition of $L^2(\Omega, \sigma(Z_t, t \geq 0), \mathbb{P})$ as

$$\bigoplus_{n=0}^{\infty} \bigoplus_{i_1,\ldots,i_n \in \mathbf{N}} \mathcal{H}^{(i_1,\ldots,i_n)},$$

where $\mathcal{H}^{(i_1,\ldots,i_n)}$ are spaces of multiple stochastic integrals with respect to strongly orthogonal Teugels martingales $Y_t^{(i)}$, $t \geq 0$. Such martingales are obtained by applying orthogonal polynomials to powers of jumps of Z_t, $t \geq 0$.
Orthogonal polynomials related to a selfdecomposable Lévy process Z_t, $t \geq 0$ can be given explicitely. These are orthogonal polynomials of $L^2([0,1], x\mathrm{d}x)$,

$$p_n(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{n+k+1}{n} x^k .$$

$p_n(x) = P_n^{(0,1)}(2x-1) \leftarrow$ Jacobi polynomial.

$$p_0(x) = 1 \qquad p_1(x) = 3x - 2$$
$$p_2(x) = 10x^2 - 12x + 3 \qquad p_3(x) = 35x^3 - 60x^2 + 30x - 4$$
$$p_4(x) = 126x^4 - 280x^3 + 210x^2 - 60x + 5$$

$$\int_0^1 p_n(x)^2 \, x\mathrm{d}x = \frac{1}{2(n+1)} .$$

$\{\sqrt{2(n+1)}\, p_n : n \geq 0\}$ is a CONS for $L^2([0,1], x\mathrm{d}x)$.

Transformation V of $\mathbb{R}$, corresponding to the shift on $\mathbb{R}^{\mathbb{Z}}$, generates an isometry on each chaos space $\mathcal{H}^{(i_1,\dots,i_n)}$. Ergodic decomposition of V induces related ergodic decompositions in the space of chaos of selfdecomposable processes.

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Thank you!