Generalized fractional Lévy processes [1]

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[1] joint work with Muneya Matsui, Keio University, Japan
partly extends work by Tina Marquardt; cf. also joint work with Holger Fink

Motivation from stochastic volatility models

As limit models, for standard Brownian motion B and fractional Brownian motion B^H ($H \in (0, 1/2)$), we consider types of models

$$\begin{aligned} dz(t) &= (\mu + \beta v(t))dt + \sqrt{v(t)}dB(t), \\ v(t) &= f(y(t)) \text{ for } f : \mathbb{R} \to \mathbb{R}_+ \text{ and} \\ dy(t) &= -\lambda y(t)dt + dB^H(t), \quad \lambda > 0, \end{aligned}$$

and

$$\begin{aligned} dz(t) &= (\mu + \beta v(t))dt + v(t)dB(t) \\ d(\log v(t)) &= -\lambda \log v(t)dt + \sigma dB^H(t). \end{aligned}$$

Generalized fractional Lévy processes

Throughout: L is a two-sided centered, finite variance Lévy process, no Gaussian component and Lévy measure ν.
W.l.o.g. : $E[(L(t))^2] = tE[(L(1))^2] = t\int_{\mathbb{R}} x^2\nu(dx) = t$ for all $t \geq 0$.
Recall, $E[\exp\{i\theta L(t)\}] = \exp\{t\psi(\theta)\}$ for $t \geq 0$, where

$$\psi(\theta) = \int_{\mathbb{R}} (e^{i\theta x} - 1 - i\theta x)\nu(dx)\,, \quad \theta \in \mathbb{R}\,. \tag{1}$$

Definition Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ with $g(0) = 0$ and such that $\int_{\mathbb{R}}\{g((t-s)_+) - g((-s)_+)\}^2 ds < \infty$ for all $t \in \mathbb{R}$. Then

$$S(t) \quad = \quad \int_{\mathbb{R}} \{g((t-u)_+) - g((-u)_+)\}dL(u)\,, \quad t \in \mathbb{R}\,, \tag{2}$$

is called generalized fractional Lévy process (GFLP). □
Example Define $g(t) = t^{H-\frac{1}{2}}$, then S is a fractional Lévy process. □

For $x > 0$ let $\sigma^2(x) := \mathrm{Var}[S(x)]$ and define the **time scaled GFLP**

$$S_x(t) := \frac{S(xt)}{\sigma(x)}, \quad t \in \mathbb{R}.$$

Define

$$\tilde{\Gamma}_x(s,t) = \frac{\mathrm{Cov}[S(xs), S(xt)]}{\sigma^2(x)}, \quad s,t > 0 .$$

Theorem Let S be as in (2) with derivative $g' \in RV_{\rho-1}$ for $\rho \in (0, \frac{1}{2})$ and B^H FBM with $H = \rho + 1/2$. Then

$$S_x \xrightarrow{d} B^H \quad \text{as } x \to \infty,$$

where convergence holds in the Skorokhod space $D(\mathbb{R})$ with the metric of uniform convergence on compacta. □

Stochastic integrals with respect to a GFLP

Throughout: $g : \mathbb{R}_+ \to \mathbb{R}_+$ has derivative $g' > 0$.
Note that for $t > 0$

$$g((t-u)_+) - g((-u)_+) = \int_{\mathbb{R}} \mathbf{1}_{(0,t]}(v) g'((v-u)_+) dv\,, \quad u \in \mathbb{R}\,.$$

Use g' as extension of the Riemann-Liouville kernel function and define for appropriate functions h

$$(I_-^g h)(u) := \int_u^\infty h(v) g'(v-u) dv = \int_{\mathbb{R}} h(v) g'((v-u)_+) dv\,, \quad u \in \mathbb{R}\,.$$

Proposition Let $g' > 0$ and $\int_0^1 g'(s)ds + \int_1^\infty (g'(s))^2 ds < \infty$. Define

$$\widetilde{\mathcal{H}} := \left\{ h : \mathbb{R}_+ \to \mathbb{R} : \int_{\mathbb{R}} (I_-^g h)^2(u) du < \infty \right\}.$$

If $h \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then $h \in \widetilde{\mathcal{H}}$.

The space $\mathcal{H}$

Define $\mathcal{H}$ as the completion of $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ wrt the norm

$$||h||_{\mathcal{H}} := \left(E[(L(1))^2] \int_{\mathbb{R}} (I_-^g h)^2(u)du\right)^{1/2}.$$

Theorem Let S be a GFLP with kernel function g and let $h \in \mathcal{H}$. Then in the $L^2(\Omega)$ sense:

$$\int_{\mathbb{R}} h(u)dS(u) = \int_{\mathbb{R}} (I_-^g h)(u)dL(u)\,.$$

Moreover, the following isometry holds:

$$||\int_{\mathbb{R}} h(u)dS(u)||^2_{L^2(\Omega)} = ||h||^2_{\mathcal{H}}\,.$$

□

Note:

$L^2(\Omega)$ and $\mathcal{H}$ are inner product spaces and for $h_1, h_2 \in \mathcal{H}$,

$$\left\langle \int_{\mathbb{R}} h_1(u)dS(u), \int_{\mathbb{R}} h_2(u)dS(u) \right\rangle_{L^2(\Omega)} = \langle h_1, h_2 \rangle_{\mathcal{H}} .$$

Proposition Let S be a GFLP with kernel function g. Then for $h_1, h_2 \in \mathcal{H}$,

$$\mathrm{Cov}\Big[\int_{\mathbb{R}} h_1(u)dS(u) \, , \int_{\mathbb{R}} h_2(v)dS(v) \Big] \quad = \quad \langle h_1, h_2 \rangle_{\mathcal{H}} .$$

Now,

$$\Gamma(u, v) = \frac{\partial^2 \mathrm{Cov}[S(u), S(v)]}{\partial u \partial v} = \int_{\mathbb{R}} g'((u-w)_+) g'((v-w)_+) dw \, ,$$

and, in particular,

$$\langle h_1, h_2 \rangle_{\mathcal{H}} = \int_{\mathbb{R}} \int_{\mathbb{R}} h_1(u) h_2(v) \int_{\mathbb{R}} g'((u-w)_+) g'((v-w)_+) dw du dv \, .$$

Fractional Ornstein-Uhlenbeck type processes

Definition Let B^H be FBM for $0 < H < 1$ and let $\lambda, \sigma > 0$.

(i) For an initial finite random variable $Y(0)$ a **fractional Ornstein-Uhlenbeck process (FOU)** is defined as

$$Y^H(t) := e^{-\lambda t}\left(Y^H(0) + \sigma \int_0^t e^{\lambda s} dB^H(s)\right), \quad t \in \mathbb{R}.$$

(ii) If the initial random variable is given by

$$Y^H(0) = \sigma \int_{-\infty}^0 e^{\lambda s} dB^H(s),$$

the FOU is **stationary** and we denote this stationary version by

$$\overline{Y}^H(t) = \sigma \int_{-\infty}^t e^{-\lambda(t-s)} dB^H(s), \quad t \in \mathbb{R}.$$

Definition Let S be a GFLP and let $\lambda, \sigma > 0$.
(i) For finite $V(0)$ an **OU process driven by a GFLP** is defined as

$$V(t) := e^{-\lambda t}\left(V(0) + \sigma \int_0^t e^{\lambda u} dS(u)\right), \quad t \in \mathbb{R}.$$

(ii) If the initial random variable is given by

$$V(0) = \sigma \int_{-\infty}^0 e^{\lambda u} dS(u), \quad t \in \mathbb{R},$$

the OU process driven by a GFLP is **stationary** and we denote this stationary version by

$$\overline{V}(t) = \sigma \int_{-\infty}^t e^{-\lambda(t-u)} dS(u), \quad t \in \mathbb{R}.$$

□

Proposition Let S be a GFLP and $\lambda > 0$. For all $t \in \mathbb{R}$, in $L^2(\Omega)$,

$$\overline{V}(t) := \int_{-\infty}^{t} e^{-\lambda(t-u)} dS(u) = \int_{-\infty}^{t} (I_{-}^{g} e^{-\lambda(t-\cdot)})(u) dL(u).$$

Furthermore, for all $s, t \in \mathbb{R}$ we have $E[\overline{V}(t)] = 0$ and

$$\text{Cov}[\overline{V}(s), \overline{V}(t)] = \int_{-\infty}^{t} \int_{-\infty}^{s} e^{-\lambda(t-u)} e^{-\lambda(s-v)} \Gamma(u, v) du dv\,,$$

where, we recall,

$$\Gamma(u, v) = \frac{\partial^2 \text{Cov}[S(u), S(v)]}{\partial u \partial v} = \int_{\mathbb{R}} g'((u-w)_+) g'((v-w)_+) dw\,.$$

The chf of $\overline{V}(t_1), \ldots, \overline{V}(t_m)$ for $t_1 < \cdots < t_m$ is

$$E\Big[\exp\Big\{\sum_{j=1}^{m} i\theta_j \overline{V}(t_j)\Big\}\Big] = \exp\Big\{\int_{\mathbb{R}} \psi\Big(\sum_{j=1}^{m} \theta_j \int_{-\infty}^{t_j} e^{-\lambda(t_j-v)} g'((v-s)_+) dv\Big) ds\Big\}\,,$$

where $\theta_j \in \mathbb{R}$, $j = 1, \ldots, m$, and ψ is given in (1).

Limit theory for OU processes driven by time scaled GFLPs

For $x > 0$ let $\sigma^2(x) := \mathrm{Var}[S(x)]$ and recall the **time scaled GFLP**

$$S_x(t) := \frac{S(xt)}{\sigma(x)}, \quad t \in \mathbb{R},$$

and note that for $x > 0$

$$S(xt) = \int \mathbf{1}_{(0,tx]}(v)dS(v) = \int_{\mathbb{R}} (I_-^g \mathbf{1}_{(0,tx]})(u)dL(u), \quad t \geq 0.$$

Theorem Let S be a GFLP with kernel function g and let $h \in \mathcal{H}$.
(i) Then for $x > 0$, in the $L^2(\Omega)$ sense,

$$\int_{\mathbb{R}} h(u)dS_x(u) = \frac{x}{\sigma(x)} \int_{\mathbb{R}} \int_{\mathbb{R}} h(v)g'((xv-u)_+)dvdL(u)\,.$$

(ii) Assume that $h_t \in \mathcal{H}$ for $t \in \mathbb{R}$. Then

$$\mathrm{Cov}\Big[\int h_s(u)dS_x(u), \int h_t(u)dS_x(u)\Big] \;=\; \int h_t(u)h_s(v)\Gamma_x(u,v)dudv\,,$$

where

$$\Gamma_x(u,v) = \frac{x^2}{\sigma^2(x)} \int g'(x(u-w)_+)g'(x(v-w)_+)dw\,.$$

Theorem If $g' \in RV_{\rho-1}$ for $\rho \in (0, \frac{1}{2})$ and $H = \rho + \frac{1}{2}$, then for $s, t \in \mathbb{R}$,

$$\begin{aligned}
\lim_{x\to\infty} \Gamma_x(s,t) &= \lim_{x\to\infty} \frac{x^2 \int_{\mathbb{R}} g'(x(s-w)_+)g'(x(t-w)_+)dw}{\sigma^2(x)} \\
&= \frac{\rho^2 \int_{\mathbb{R}} (s-w)_+^{\rho-1}(t-w)_+^{\rho-1} dw}{\int_{\mathbb{R}} \{(1-u)_+^{\rho} - (-u)_+^{\rho}\}^2 du} \\
&= \frac{\partial^2}{\partial t \partial s} \mathrm{Cov}(B^H(t), B^H(s)) = H(2H-1)|t-s|^{2H-2} \\
\lim_{x\to\infty} \widetilde{\Gamma}_x(s,t) &= \frac{\int_{\mathbb{R}} \{(s-u)_+^{\rho} - (-u)_+^{\rho}\}\{(t-u)_+^{\rho} - (-u)_+^{\rho}\} du}{\int_{\mathbb{R}} \{(1-u)_+^{\rho} - (-u)_+^{\rho}\}^2 du} \\
&= \mathrm{Cov}(B^H(s), B^H(t)) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}).
\end{aligned}$$

□

Note that $E[S_x(t)] = 0$ and $\mathrm{Var}[S_x(t)] = 1$ for all $t \in \mathbb{R}$.

Definition For $\lambda, \sigma > 0$ we define the **OU process driven by the time-scaled GFLP**

$$V_x(t) := e^{-\lambda t}\left(V_x(0) + \sigma \int_0^t e^{\lambda s} dS_x(s)\right), \quad t \geq 0.$$

(ii) If the initial random variable is given by

$$V_x(0) = \sigma \int_{-\infty}^0 e^{\lambda s} dS_x(s),$$

then V_x is stationary and we denote this stationary process by

$$\overline{V}_x(t) := \sigma \int_{-\infty}^t e^{-\lambda(t-s)} dS_x(s), \quad t \in \mathbb{R}.$$

□

Proposition For $x > 0$ let S_x be a time-scaled GFLP.
(i) For $t \in \mathbb{R}$, in the $L^2(\Omega)$-sense,

$$\overline{V}_x(t) = \int_{-\infty}^{t} e^{-\lambda(t-u)}(u)dS_x(u) = \frac{x}{\sigma(x)} \int_{\mathbb{R}} \int_{-\infty}^{t} e^{-\lambda(t-v)} g'((xv-u)_+)dvdL(u).$$

(ii) For $s, t \in \mathbb{R}$, we have $E[\overline{V}_x(t)] = 0$ and

$$\mathrm{Cov}[\overline{V}_x(s), \overline{V}_x(t)] = \int_{-\infty}^{s} \int_{-\infty}^{t} e^{-\lambda(t-u)}(u)e^{-\lambda(s-v)}\Gamma_x(u, v)dudv,$$

where

$$\Gamma_x(u, v) = \frac{x^2}{\sigma^2(x)} \int_{\mathbb{R}} g'(x(u-w)_+)g'(x(v-w)_+)dw.$$

□

Theorem Let S be a GFLP with kernel function g and $g' \in RV_{\rho-1}$ for $\rho \in (0, \frac{1}{2})$. For $x > 0$, recall that

$$\overline{V}_x^{\rho}(t) = \sigma \int_{-\infty}^{t} e^{-\lambda(t-s)} dS_x(s)\,, \quad t \in \mathbb{R}\,,$$

is a stationary OU process driven by the time-scaled GFLP. Then

$$\overline{V}_x(\cdot) \xrightarrow{d} \overline{Y}^H(\cdot) = \sigma \int_{-\infty}^{\cdot} e^{-\lambda(t-s)} dB^H(s) \text{ as } x \to \infty,$$

where convergence holds in $D(\mathbb{R})$ with the metric of uniform convergence on compacta. Recall that $H = \rho + \frac{1}{2}$. □

We show joint weak convergence of price and volatility process to a continuous time long memory stochastic volatility model.

Theorem [Jacod and Shiryaev]

For $x > 0$ we consider, with $H = \rho + \frac{1}{2}$,

$$\begin{aligned} z_x(t) &:= \mu t + \beta \int_0^t v_x(s)ds + \int_0^t \sqrt{v_x(s)}dB_s, \\ v_x(t) &:= f(\overline{V}_x^\rho(t)), \end{aligned}$$

where $f : \mathbb{R} \to \mathbb{R}_+$ is a continuous function. Then for $x \to \infty$ the bivariate process $(z_x(\cdot), v_x(\cdot))$ converges in $D(\mathbb{R}_+^2)$, with the mtric of uniform convergence on compacta, to

$$\begin{aligned} z(t) &:= \mu t + \beta \int_0^t v(s)ds + \int_0^t \sqrt{v(s)}dB_s, \\ v(t) &:= f(\overline{Y}^H(t)). \end{aligned}$$